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Diagnosable evaluation of DCC linear congruential graphs under the PMC diagnostic model

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ABSTRACT

A DCC (disjoint consecutive cycles) linear congruential graph $G(F,n)$ consists of n nodes and is generated by a set of linear functions F with special properties. It was proved that $G(F,n)$ is a $2t$-regular graph and has connectivity $2t$, where $t = |F|$ and $1 \leqslant t \leqslant p-1$ ($n = 2^p$ for some integer p). For a multiprocessor system, its diagnosability is critical to measure the performance. In this paper, we study the diagnosability of $G(F,2^p)$ under the precise and pessimistic diagnosis strategies based on the PMC (Preparata, Metze, and Chien) diagnostic model. It is proved that $G(F,2^p)$ is $2t$-diagnosable and $(4t-5)/(4t-5)$-diagnosable under the two diagnosis strategies, respectively, where $p \geqslant 3$ and $2 \leqslant t \leqslant p-1$. In addition, the diagnosability of DCC linear congruential graphs is compared with that of BC (bijective connection) graphs.

1. Introduction

In a multiprocessor system, there are normally hundreds of thousands of processors working simultaneously on a problem at a very high speed. Thus, it is unavoidable that the processors in the system become faulty. In order to solve this problem, the faulty processors should be identified and then replaced by the fault-free processors. Therefore, fault diagnosis in the multiprocessor system has become more and more important. In the literature, the process of identifying the faulty processors is called the *diagnosis* of the system and the maximal number of faulty nodes that a system can guarantee to diagnose is called the *degree of diagnosability* of the system.

So far, a number of system-level diagnosis strategies have been proposed [6,11–14] and many of them are based on the PMC diagnostic model proposed by Preparata, Metze, and Chien [1,8,24,25]. In the PMC diagnostic model, a self-diagnostic system is represented by a digraph $G = (V,A)$, where (1) V represents the set of nodes, each of which represents a processor; (2) A the set of arcs in G, each of which represents a test link that u can test v. Here, the test-result is a function f: $A \rightarrow \{0,1\}$ such that if v is tested to be fault-free by u then $f(\langle u,v\rangle) = 0$; otherwise $f(\langle u,v\rangle) = 1$. The PMC diagnostic model assumes that a fault-free node should always give correct test-result, whereas the test-result given by a faulty node is unreliable. That is,

Definition 1. A *syndrome* is a function s: $A \rightarrow \{0,1\}$. A fault-set $F \subseteq V$ is called to be *consistent* with s if following two conditions are satisfied:

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(1) $v \in V - F$ if $u \in V - F$ and $f(\langle u,v\rangle) = 0$;
(2) $v \in F$ if $u \in V - F$ and $f(\langle u,v\rangle) = 1$.

As a representative diagnosis strategy based on the PMC diagnostic model [18], *the precise strategy* [22] defines the diagnosability of a system as follows:

Definition 2 [18]. A system is said to be t_p-*diagnosable* if for any syndrome s, there exists at most one fault-set $F \subseteq V$ that is consistent with s, given that $|F| \leqslant t_p$.

Hakimi and Amin characterized the testing assignments such that the fault-set F can be uniquely identified on the basis of any given collection of test-results, provided that the number of faulty nodes does not exceed the given bound t_p [12].

Another representative diagnosis strategy based on the PMC diagnostic model was proposed by Friedman [11] and is called the *pessimistic strategy* [16], which defines the diagnosability of a system as follows.

Definition 3 [11]. A system is said to be t_1/t_1-*diagnosable* if, for any syndrome s, a fault-set $F' \subseteq V$ can be identified such that $|F'| \leqslant t_1$, and if $v \in V$ is faulty, then $v \in F'$, provided that the number of faulty nodes in the system does not exceed t_1.

Chwa and Hakimi characterized t_1/t_1-diagnosable systems and provided a necessary and sufficient condition for being such systems [5].

It should be pointed out that the above two representative diagnosis strategies have different features. First, the nodes in the set F located by the precise diagnosis strategy are all faulty. Therefore, all the nodes in F should be replaced in the system repair process. Second, the node subset F located by the pessimistic diagnosis strategy contains all faulty nodes, but the faulty/fault-free status of a specific node in F cannot be further identified. Thus, if all these nodes in F (some of which may be fault-free), are replaced in the system repair process, then additional cost will be paid for the replacement of the fault-free nodes in F. Fortunately, it was shown that at most one fault-free node may be contained in F [5,23]. Finally, the diagnosability of the same system under the pessimistic strategy tends to be larger than that of the system under the precise strategy. As a result, the two diagnosis strategies have merits as well as shortcomings. For a smaller system, the precise strategy is preferred while, for a larger system, the pessimistic strategy is preferred, because the additional cost paid for the replacement of the fault-free node (at most one) is negligible.

Considering that an undirected graph is just a special case of a digraph, the PMC diagnostic model can also be used in the self-diagnostic systems represented by undirected graphs. The topology of a multiprocessor system may be represented by an undirected graph $G = (V,E)$, where every node $u \in V$ represents a processor and every edge $(u,v) \in E$ represents the communication link between u and v. In order to save the cost to transmit test-results, the test links are generally the same as the communication links so that tests can be carried out only between neighbors, that is, if $(u,v) \in E$, without considering the faulty/fault-free status of u and v, v can be tested with u and vice versa. Consequently, the diagnosability of a multiprocessor system can be changed into that of a self-diagnostic system represented by an undirected graph [2,4,7,8,10,15].

So far, a lot of interconnection networks have been proposed. As a representative, *BC* (*bijective connection*) *graphs* [8–10] include hypercube, crossed cube, and Möbius cube. Here, any n-dimensional BC graph is an n-regular graph with 2^n nodes, logarithm diameter and high connectivity. It was proved that any n-dimensional BC graph is n-diagnosable and $(2n-2)/(2n-2)$-diagnosable under the precise and pessimistic strategies, respectively [8]. Compared to the BC graphs, the *DCC* (*disjoint consecutive cycles*) *linear congruential graphs* introduced by Opatrny et al. [17], are also regular graphs. However, the node number of a DCC linear congruential graph is not only confined to a power of 2 and its node degree may change in a large range even if its node number is fixed (for example, a power of 2). That is, the topological structure of a DCC linear congruential graph is changeable when its node number is fixed, whereas the topological structure of a BC graph is uniquely defined by its node number. In [17], some properties of DCC linear congruential graphs were studied, particularly when their node numbers are powers of 2. In this paper, we study the diagnosability of DCC linear congruential graphs under the two above-mentioned diagnosis strategies. It is proved that a DCC linear congruential graph with 2^p nodes is $2t$-diagnosable and $(4t-5)/(4t-5)$-diagnosable under the precise and pessimistic strategies, respectively, where $2t$ is its node degree or node connectivity.

The rest of this paper is organized as follows. Section 2 gives some definitions and notations. Then, the diagnosability of DCC linear congruential graphs under the precise and pessimistic diagnosis strategies are studied in Section 3. Finally, we conclude the paper in Section 4.

2. Preliminaries

In this section, we first give some relevant definitions and notations in the graph theory.

If $V_1 \subseteq V(G)$, then we use $G - V_1$ to denote the induced subgraph of $V(G) - V_1$ in G and the connectivity of G is denoted by $\kappa(G)$. In this paper, isomorphic graphs may be considered as an identical graph. For $V', V'' \subseteq V(G), V' \bigcap V'' = \emptyset, V' \neq \emptyset$, and $V'' \neq \emptyset$, we define the test-set of V' in G to be $\Gamma(G,V') = \{v \in V(G) - V'|$. There exists a node $v' \in V'$ such that $(v,v') \in E(G)\}$. The edge set between V' and V'' is denoted by $[V',V''] = \{(v',v'') \in E(G) | v' \in V'$ and $v'' \in V''\}$. The neighbor-set for V' in V'' is defined as $N(V'',V') = \{x \in V''|$ There exists a node $y \in V'$ such that $(x,y) \in E(G)\}$. Obviously, $\Gamma(G,V') = N(V(G) - V', V')$.

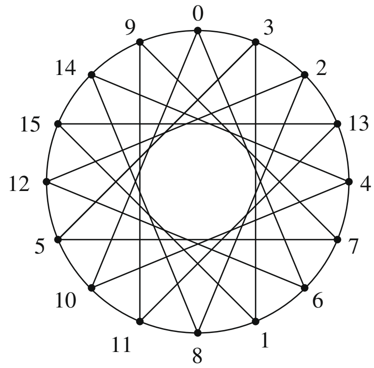

Fig. 1. A DCC linear congruential graph $G(\{f_1, f_2\}, 16)$, where $f_1(x) = 5x + 3$ and $f_2(x) = 9x + 6$.

In this paper, N denotes the set of all the nonnegative integers. For two integers m and n, if n is a multiple of m, then we write $m|n$; otherwise we write $m \nmid n$. The greatest common divisor of m and n is denoted by $\gcd(m,n)$.

Then, we formally define DCC linear congruential graphs as follows:

Definition 4 [17]. Let n be a positive integer and $F = \{f_i(x) = a_i x + c_i | 1 \leqslant i \leqslant t$, where $a_i, c_i \in N\}$ a finite set of t linear functions for some integer t. Define a DCC linear congruential graph $G(F, n)$ as the graph on the node set $\{0, 1, \ldots, n-1\}$, in which any $x \in V$ is adjacent to $f_i(x) \bmod n$, for every integer i with $1 \leqslant i \leqslant t$.

It should be pointed out that all the mentioned graphs are simple, connected, undirected graphs. For an example of a DCC linear congruential graph, please see Fig. 1.

3. t_p-Diagnosability and t_1/t_1-diagnosability of DCC linear congruential graphs

In this section, we will study t_p-diagnosability and t_1/t_1-diagnosability of DCC linear congruential graphs. We first introduce some of relevant results associated with DCC linear congruential graphs in [17].

Lemma 5 [17]. *Let n be a positive integer that contains at least one multiple factor, i.e., $n = k^p m$ for some integers $k > 1$, $p \geqslant 2$, and m. Let c be an integer such that $\gcd(c, n) = 1$. Let b be a multiple of every prime factor of n; b is also a multiple of 4, if n is a multiple of 4. For any i, $1 \leqslant i \leqslant p+1$, let $f_i(x) = (k^{i-1}b + 1)x + k^{i-1}c$.*

Then, the function f_i generates k^{i-1} node-disjoint cycles of length $\frac{n}{k^{i-1}}$ on the set $\{0, 1, \ldots, n-1\}$. The node sets of these cycles are the sets

$$A_{1i} = \{0, k^{i-1}, 2k^{i-1}, \ldots, n - k^{i-1}\},$$
$$A_{2i} = \{1, k^{i-1} + 1, 2k^{i-1} + 1, \ldots, n - k^{i-1} + 1\},$$
$$\ldots$$
$$A_{k^{i-1}i} = \{k^{i-1} - 1, 2k^{i-1} - 1, 3k^{i-1} - 1, \ldots, n-1\}.$$

Furthermore, there is an edge between two nodes x and y in the graph generated by f_i only if $k^{i-1} | (y - x)$ but $k^i \nmid (y - x)$.

Notation 6. For $x, y \in \{0, 1, \ldots, n-1\}$ and $1 \leqslant i \leqslant t$, if $y = f_i(x) \bmod n$, we write $x \overset{i}{\text{——}} y$ and say x is adjacent to y with respect to f_i.

Theorem 7 [17]. *Let n be a positive integer that contains at least one multiple factor, i.e., $n = k^p m$ for some integers $k > 1$, $p \geqslant 2$, and m. For an integer t in $\{1, 2, \ldots, p+1\}$, let F be a set of t linear functions such that:*

(1) *each function in F is of cycle type k^j on $\{0, 1, \ldots, n-1\}$ for some j, $0 \leqslant j \leqslant p$, such that $k^j < \frac{n}{2}$;*
(2) *there is exactly one function in F of cycle type 1 on the set $\{0, 1, \ldots, n-1\}$;*
(3) *any two functions in F are of different cycle types on $\{0, 1, \ldots, n-1\}$. Then the graph $G(F, n)$ is a regular and connected graph of degree $2t$.*

Definition 8. Let n be a positive integer that contains at least one multiple factor, i.e., $n = k^p m$ for some integers $k > 1$, $p \geqslant 2$, and m. For any given integer $t \in \{1, 2, \ldots, p+1\}$ such that $k^{i-1} < \frac{n}{2}$, we say that a set F of t linear functions is a disjoint consecutive cycles set (DCC set for short) with respect to the integer n if for each j, $0 \leqslant j \leqslant t-1$, there is exactly one function in F of cycle type k^j on the set $\{0, 1, \ldots, n-1\}$.

Definition 9. Let n be a positive integer that contains at least one multiple factor, i.e., $n = k^p m$ for some integers $k > 1$, $p \geqslant 2$, and m. For any given $t \in \{1, 2, \ldots, p+1\}$ such that $k^{t-1} < \frac{n}{2}$, we define a DCC linear congruential graph $G_{2t}(F, n)$ as a linear congruential graph $G(F, n)$ generated by DCC set F of t linear functions with respect to n.

By Theorem 7, we have

Corollary 10. *Let n be a positive integer that contains at least one multiple factor, i.e., $n = k^p m$ for some integers $k > 1$, $p \geqslant 2$, and m. For any given integer $t \in \{1, 2, \ldots, p+1\}$ such that $k^{t-1} < \frac{n}{2}$, let F be a DCC set of t linear functions. Then the graph $G_{2t}(F, n)$ is a regular and connected graph of degree $2t$.*

When $k = 2$ and $m = 1$, Opatrny et al. proved the following result:

Corollary 11 [17]. *Let $G_{2t}(F, n)$ be a DCC linear congruential graph on the node set $\{0, 1, n-1\}$, where $F = \{f_i | 1 \leqslant i \leqslant t\}$, with $f_i(x) = a_i x + c_i$ verifying the conditions given above. Let $V_1 = \{0, 2, \ldots, 2^p - 2\}$ and $V_2 = \{1, 3, \ldots, 2^p - 1\}$. Then, the subgraphs $G[V_1]$ and $G[V_2]$ are isomorphic to the DCC linear congruential graphs $G_{2t-2}(F', 2^{p-1})$ and $G_{2t-2}(F'', 2^{p-1})$, where $F' = \{f'_i \mid 1 \leqslant i \leqslant t-1\}$ and $F'' = \{f''_i | 1 \leqslant i \leqslant t-1\}$ with $f'_i(x) = a_{i+1}x + \frac{c_{i+1}}{2}$ and $f''_i(x) = a_{i+1}x + \frac{c_{i+1} + a_{i+1} - 1}{2}$, respectively. Moreover, G is the edge-disjoint union of $G[V_1], G[V_2]$, and of the Hamiltonian cycle induced by f_1 which forms a bipartite graph of degree 2 on (V_1, V_2).*

Theorem 12 [17]. *Let $t \leqslant p$ be an integer and $F = \{f_i \mid 0 \leqslant i \leqslant t-1\}$ be defined as above. Then the DCC linear congruential graph $G_{2t}(F, 2^p)$ is 2t-connected.*

By above results and definitions, we have

Corollary 13. *Let $t \leqslant p$ be an integer and $F = \{f_i \mid 0 \leqslant i \leqslant t-1\}$ be defined as above. Then the DCC linear congruential graph $G_{2t}(F, 2^p)$ is a simple graph and $\kappa(G_{2t}(F, 2^p)) = 2t$.*

So far, we have introduced some of relevant definitions and results in [17]. Now, we discuss the diagnosability of the DCC linear congruential graph $G_{2t}(F, n)$ under the precise and pessimistic strategies, where $F = \{f_i \mid 1 \leqslant i \leqslant t\}$, with $f_i(x) = a_i x + c_i$ verifying the conditions in Corollary 11.

Lemma 14 ([12,18]). *Let the graph $G = (V, E)$ be the representation of a system S, with V representing the processors in S and E the interconnection among them. Then, the sufficient conditions for S to be t_p-diagnosable are $|V| \geqslant 2t_p + 1$ and $\kappa(G) \geqslant t_p$; the necessary conditions for S to be t_p-diagnosable are $|V| \geqslant 2t_p + 1$ and each processor is tested by at least t_p other processors.*

According to Lemma 14 and Corollary 13, we have

Theorem 15. *Let $p > 4$ and $1 \leqslant t \leqslant p-1$. Then $G_{2t}(F, 2^p)$ is 2t-diagnosable.*

Lemma 16 [5]. *Let the graph $G = (V, E)$ be the representation of a system S, with V representing S's processors and E the interconnection among them. Then, S is t_1/t_1-diagnosable under the pessimistic strategy if and only if for any integer k with $1 \leqslant k \leqslant t_1$ and any $V' \subset V$ such that $|V'| = 2k, |\Gamma(G, V')| \geqslant t_1 - k + 1$.*

In the following, we will prove that $G_{2t}(F, 2^p)$ is $(4t-5)/(4t-5)$-diagnosable for $p \geqslant 3$ and $2 \leqslant t \leqslant p-1$.

Lemma 17 [7]. *Let G be a connected graph, $V' \subset V(G)$ and $V'' = V(G) - V'$. Then*

(1) *If $|V''| \geqslant \kappa(G)$, then $|N(V'', V')| \geqslant \kappa(G)$;*
(2) *If $|V''| < \kappa(G)$, then $|N(V'', V')| = |V''|$.*

In what follows, recall that $V_1 = \{0, 2, \ldots, 2^p - 2\}$ and $V_2 = \{1, 3, \ldots, 2^p - 1\}$.

Lemma 18. *Let $p \geqslant 3$ and $2 \leqslant t \leqslant p-1$. Then, for any $V' \subset V_1$ such that $|V'| \geqslant 2, |N(V_2, V')| \geqslant |V'|$.*

Proof. We prove this lemma by contradiction. If $|N(V_2, V')| = q < |V'|$, by Lemma 5 and Corollary 11, $|N(V_2, V')| = \{x \in V_2|$ There exists a node $y \in V'$ such that $x \overset{1}{\text{—}} y\}$, and $|[V', V_2]| = 2|V'| > 2q$.

If $|V'| > 2$, there exists a node $u \in N(V_2, V')$ such that u is adjacent to at least three nodes in V' with respect to f_1. This is contradictory to Corollary 11.

If $|V'| = 2$, there exist multiple edges in $G_{2t}(F, 2^p)$. This is contradictory to Corollary 13. □

Lemma 19. *Let $p \geqslant 3$ and $2 \leqslant t \leqslant p-1$, and $G = G_{2t}(F, 2^p)$. Then, for any $u, v \in V_1$ such that $u \neq v$ and $|N(V_2, \{u, v\})| = 3, |\Gamma(G, \{u, v\})| \geqslant 4t - 5$.*

Proof. Let $n = 2^p$. Since both u and v are adjacent to two nodes in V_2 with respect to f_1 and $|[N(V_2, \{u, v\})| = 3$, by Corollary 11, there exists a node $w \in V_2$ such that w is adjacent to both u and v with respect to f_1. Without loss of generality, we let $w = f_1(u) \bmod n$ and $v = f_1(w) \bmod n$. That is, $w = (b+1)u + c \bmod n$ and $v = (b+1)w + c \bmod n$. Then, $v = (b+1)^2 u + (b+2)c \bmod n$. Let $A = \{y \in V_1 | u \overset{2}{\text{—}} y\}, B = \{y \in V_1 | u \overset{i}{\text{—}} y, 2 < i \leqslant t\}, C = \{y \in V_1 | v \overset{2}{\text{—}} y\}$. Then, $|A| = |C| = 2$, $N(V_1 - \{u\}, \{u\}) = A \bigcup B, A \bigcap B = \emptyset$, and $C \subseteq N(V_1 - \{v\}, \{v\})$.

Let $x \in B - \{v\}$. Then $x \overset{i}{\text{—}} u$ for some i, $2 < i \leqslant t$. By Lemma 5, $2^{i-1}|(u-x)$ but $2^i \dagger (u-x)$. Thus $4 \dagger (u-x)$. Since $v - x = (b(b-2)u + (u-x) + bc) + 2c \bmod n$, $2|(v-x)$ but $4 \dagger (v-x)$. Hence, if there exists an integer j with $2 \leqslant j \leqslant t$ such that $v \overset{j}{\text{—}} x$, then by Lemma 5, $j = 2$. Furthermore, considering that $|C| = 2$, there exist at most two nodes in V_1 which are adjacent to both u and v. There are the following two cases:

Case 1. $(u, v) \notin E(G)$. Then, there exist at most four nodes in V_1 (two out of which belong to A and the other two out of which belong to C) which are adjacent to both u and v. Thus, $|\Gamma(G, \{u, v\})| \geqslant |\Gamma(G[V_1], \{u, v\})| + |N(V_2, \{u, v\})| \geqslant (4(t-1) - 4) + 3 = 4t - 5$.

Case 2. $(u, v) \in E(G)$. Then, $u \overset{i}{\text{—}} v$ for some i, $2 \leqslant i \leqslant t$. Since $(v-u) = ((b^2 + 2b)u + bc) + 2c$, $2|(v-u)$ but $4 \dagger (v-u)$. By Lemma 5, $i = 2$. Furthermore, since $|(A \bigcup C) - \{u, v\}| \leqslant 2$, there exist at most two nodes in $V_1 - \{u, v\}$ (out of $(A \bigcup C) - \{u, v\}$) which are adjacent to both u and v. Thus, $|\Gamma(G, \{u, v\})| \geqslant |\Gamma(G[V_1], \{u, v\})| + |N(V_2, \{u, v\})| \geqslant (4(t-1) - 2 - |\{u, v\}|) + 3 = 4t - 5$. □

In the following theorem, we prove that $G = G_{2t}(F, 2^p)$ satisfies the conditions in Lemma 16 when $t = 2$.

Lemma 20. *Let $p \geqslant 3$ and $t = 2$, and $G = G_{2t}(F, 2^p)$. Then, for any integer k with $1 \leqslant k \leqslant 4t - 5$ and any $V' \subset V(G)$ such that $|V'| = 2k, |\Gamma(G, V')| \geqslant (4t-5) - k + 1$.*

Proof. If $p = 3$ and $k = 4t - 5 = 3$, then by Corollary 13, $|V(G) - V'| = 2^p - 2k = 2 < \kappa(G) = 4$. By Lemma 17, $|\Gamma(G, V')| = |N(V(G) - V', V')| = |V(G) - V'| = 2 > (4t-5) - k + 1$.

If $p \geqslant 3$ or $1 \leqslant k \leqslant 4t - 5 = 3$, then $|V(G) - V'| \geqslant 4 = \kappa(G)$. By Lemma 17, $|\Gamma(G, V')| = |N(V(G) - V', V')| \geqslant \kappa(G) = 4 > (4t-5) - k + 1$. □

Next, we prove that $G = G_{2t}(F, 2^p)$ satisfies the conditions in Lemma 16 when $k = 1$.

Lemma 21. *Let $p \geqslant 3$ and $2 \leqslant t \leqslant p - 1$, and $G = G_{2t}(F, 2^p)$. Then, for any $V' \subset V(G)$ such that $|V'| = 2$, $|\Gamma(G, V')| \geqslant 4t - 5$.*

Proof. When $p \geqslant 3$ and $t = 2$, by Lemma 20, the theorem is true. When $P \geqslant 4$, for any $V' \subset V(G)$ such that $|V'| = 2$, let $V' = V'_1 \bigcup V'_2$ with $V'_1 \subset V_1$ and $V'_2 \subset V_2$. Without loss of generality, we may only consider the following cases.

Case 1. $|V'_1| = |V'_2| = 1$. By Corollary 10, $|\Gamma(G, V')| \geqslant |\Gamma(G, V'_1)| + |\Gamma(G, V'_2)| = (2t-2) + (2t-2) > 4t - 5$.

Case 2. $V'_1 = V'$. We use induction on t. When $t = 2$, by Lemma 20, the theorem is true. Suppose that the theorem holds for $\tau (\tau \geqslant 2)$. We will prove that the theorem holds for $t = \tau + 1$. By Corollary 11, let $H = G_{2\tau+2}(F, 2^p), H_1 = G_{2\tau}(F', 2^{p-1})$, and $H_2 = G_{2\tau}(F'', 2^{p-1})$. According to the induction hypothesis and Corollary 11, $|\Gamma(H_1, V')| \geqslant 4\tau - 5$.

If $|N(V_2, V')| = 4$, then $|\Gamma(H, V')| \geqslant |\Gamma(H_1, V')| + |N(V_2, V')| \geqslant (4\tau - 5) + 4 = 4(\tau + 1) - 5$.

Otherwise, by Lemma 18, $|N(V_2, V')| \geqslant |V'| = 2$. If $|N(V_2, V')| = 2$, by Corollary 11, the four nodes in $V' \bigcup N(V_2, V')$ form a cycle which is generated by f_1. Hence, $|V(H)| = 4 < 2^p$, a contradiction. Therefore, $|N(V_2, V')| = 3$. By Lemma 19, $|\Gamma(H, V')| \geqslant 4(\tau + 1) - 5$.

To sum up, the theorem holds for $t = \tau + 1$. □

Furthermore, we prove that $G_{2t}(F, 2^p)$ satisfies the conditions given in Lemma 16 when $p \geqslant 3$ and $2 \leqslant t \leqslant p - 1$.

Theorem 22. *Let $p \geqslant 3$ and $2 \leqslant t \leqslant p - 1$, and $G = G_{2t}(F, 2^p)$. Then, for any integer k with $1 \leqslant k \leqslant 4t - 5$ and any $V' \subset V(G)$ such that $|V'| = 2k, |\Gamma(G, V')| \geqslant (4t-5) - k + 1$.*

Proof. By Lemma 20, the theorem is true for $p = 3$. For $p \geqslant 4$, we use induction on t.

By Lemma 20, the theorem is true for $t = 2$.

Suppose that the theorem holds for $\tau (\tau \geqslant 2)$. By Lemma 21, we only need to prove that the theorem holds for $t = \tau + 1 (\tau + 1 \leqslant p - 1)$ and $2 \leqslant k \leqslant 4\tau - 1$. By Corollary 11, let $H = G_{2\tau+2}(F, 2^p), H_1 = G_{2\tau}(F', 2^{p-1})$, and $H_2 = G_{2\tau}(F'', 2^{p-1})$, where F' and F'' are defined like Corollary 11. For any integer k with $2 \leqslant k \leqslant 4\tau - 1$ and any $V' \subset V(H)$ such that $|V'| = 2k$, if $p = 4$, then $2 \leqslant t \leqslant 3$. Considering that $t = \tau + 1 \geqslant 3$, $t = 3$ and $\tau = 2$.

If $p = 4$ and $6 \leqslant k \leqslant 4\tau - 1 = 7$, then $2 \leqslant |V(H) - V'| = 2^p - 2k < 6 = \kappa(H)$. By Lemma 17, $|\Gamma(H, V')| = |N(V(H) - V', V')| = |V(H) - V'| \geqslant 2 \geqslant (4\tau - 1) - k + 1$.

Otherwise, we have $p = 4$ and $2 \leqslant k \leqslant 5$, or $p \geqslant 5$. For $p = 4$ and $2 \leqslant k \leqslant 5, |V(H) - V'| = 2^p - 2k \geqslant 6 = \kappa(H)$. For $p \geqslant 5$, since $2 \leqslant t \leqslant p - 1$ and $t = \tau + 1 \geqslant 3, 2 \leqslant \tau \leqslant p - 2$. Considering that $2 \leqslant k \leqslant 4\tau - 1$, we have $2 \leqslant k \leqslant 4(p-2) - 1 = 4p - 9$. Thus, $|V(H) - V'| = 2^p - 2k \geqslant 2^p - 2(4p - 9) \geqslant 2(p-2) + 2 \geqslant 2\tau + 2 = \kappa(H)$. As a result, for $p = 4$ and $2 \leqslant k \leqslant 5$, or $p \geqslant 5$, we always have $|V(H) - V'| \geqslant \kappa(H)$. Furthermore, we have the following cases.

Case 1. $2\tau - 2 \leqslant k \leqslant 4\tau - 1$. By Lemma 17, $|\Gamma(H, V')| = |N(V(H) - V', V')| = |V(H) - V'| \geqslant \kappa(H) = 2\tau + 2 \geqslant (4\tau - 1) - k + 1$.

Case 2. $2 \leqslant k \leqslant 2\tau - 3$. Then $k \leqslant 4\tau - 5$. Without loss of generality, let $V' = V'_1 \bigcup V'_2$ with $V'_1 \subset V(H_1), V'_2 \subset V(H_2)$, and $|V'_1| \geqslant |V'_2|$. Then, $2 \leqslant |V'_2| \leqslant |V'_1| < 4\tau - 5$. Since $|V'| = 2k \leqslant 4\tau - 6 \leqslant 4(p-2) - 6 < 2^{p-1}, V'_1 \subset V(H_1)$, and $V'_2 \subset V(H_2)$.

Case 2.1. $|V'_2| = 0$. By the induction hypothesis and Corollary 11, $|\Gamma(H, V'_1)| \geqslant (4\tau - 5) - k + 1$. Since $|V'_1| \geqslant 4$, by Lemma 18, $|N(V(H_2), V'_1)| \geqslant |V'_1| \geqslant 4$. Therefore, $|\Gamma(H, V')| \geqslant |\Gamma(H_1, V'_1)| + |N(V(H_2), V'_1)| \geqslant (4\tau - 1) - k + 1$.

Case 2.2. Both $|V'_2|$ and $|V'_2|$ are even, $V'_1 \neq \emptyset$, and $V'_2 \neq \emptyset$. By the induction hypothesis, $|\Gamma(H_1, V'_1)| \geqslant (4\tau - 5) - \frac{|V'_1|}{2} + 1$ and $|\Gamma(H_2, V'_2)| \geqslant (4\tau - 5) - \frac{|V'_2|}{2} + 1$. Thus, $|\Gamma(H, V')| \geqslant |\Gamma(H_1, V'_1)| + \Gamma(H_2, V'_2)| \geqslant (4\tau - 1) - k + 1$.

Case 2.3. Both $|V'_2|$ and $|V'_2|$ are odd. We deal with the following sub-cases.
Case 2.3.1. $|V'_2| = 1$. Select $x \in V'_1$. By the induction hypothesis and Corollary 11, $|\Gamma(H_1, V'_1)| \geqslant |\Gamma(H_1, V'_1 - \{x\})| - 1 \geqslant (4\tau - 5) - \frac{|V'_1|-1}{2} \geqslant (4\tau - 5) - k + 1$. By Corollary 10, $|\Gamma(H_2, V'_2)| = 2\tau$. Thus, $|\Gamma(H, V')| \geqslant |\Gamma(H_1, V'_1)| + \Gamma(H_2, V'_2)| \geqslant (4\tau - 5) - k + 1 + 2\tau \geqslant (4\tau - 1) - k + 1$.
Case 2.3.2. $|V'_2| > 1$. Similar to Case 2.3.1, we have $|\Gamma(H_1, V'_1)| \geqslant (4\tau - 5) - \frac{|V'_1|-1}{2}$ and $|\Gamma(H_2, V'_2)| \geqslant (4\tau - 5) - \frac{|V'_2|-1}{2}$. Thus, $|\Gamma(H, V')| \geqslant |\Gamma(H_1, V'_1)| + |\Gamma(H_2, V'_2)| \geqslant (4\tau - 1) - k + 1$.

In summary, the theorem holds for $t = \tau + 1$. □

By Lemma 14 and Theorem 22, we have:

Theorem 23. *If $p \geqslant 3$ and $2 \leqslant t \leqslant p - 1$, then $G_{2t}(F, 2^p)$ is $(4t - 5)/(4t - 5)$-diagnosable under the pessimistic diagnosis strategy.*

Theorem 23 shows that the degree of diagnosability of $G_{2t}(F, 2^p)$ is not less than $4t - 5$ for $p \geqslant 3$ and $2 \leqslant t \leqslant p - 1$ under the pessimistic diagnosis strategy. Furthermore, we show that $G_{2t}(F, 2^p)$ may not be t_1/t_1-diagnosable under this diagnosis strategy when $t_1 > 4t - 5$, even if $p \geqslant 3$ and $2 \leqslant t \leqslant p - 1$.

Let $p = 6$, $t = 5$, $b = 4$, $c = 1$, $F = \{f_i | 1 \leqslant i \leqslant t\}$, and $G = G_{10}(F, 64)$, where $f_i(x) = (2^{i+1} + 1)x + 2^{i-1}, 1 \leqslant i \leqslant 5$. Select $V' = \{0, 8\}$. We have $0 \xrightarrow{4} 8, 0 \xrightarrow{3} 4, 4 \xrightarrow{3} 8, 0 \xrightarrow{5} 16, 8 \xrightarrow{4} 16, 56 \xrightarrow{4} 0, 56 \xrightarrow{5} 8$. Clearly, $|\Gamma(G, \{0, 8\})| = 4t - 5$. This means that the system cannot guarantee to diagnose all the faulty nodes if there are more than $4t - 5 = 15$ faulty nodes in $G_{10}(F, 64)$.

4. Discussion and conclusions

Some properties of DCC linear congruential graphs were studied in [17]. In this paper, we focus on the diagnosability of DCC linear congruential graphs under the precise and pessimistic strategies based on the PMC diagnostic model. It is proved that the DCC linear congruential graph $G_{2t}(F, 2^p)$ is $2t$-diagnosable and $(4t - 5)/(4t - 5)$-diagnosable under the two diagnosis strategies, respectively.

In [3,7,16,22], the diagnosabilities of hypercubes, enhanced hypercubes, and Möius cubes were studied under the two diagnosis strategies. All these graphs have the similar diagnosabilities under the two diagnosis strategies. That is, each of these graphs is t_p-diagnosable and t_1/t_1-diagnosable, respectively, under the two diagnosis strategies, where t_p is the degree of the corresponding graph and t_1 doubles the degree of the corresponding graph minus a constant. As pointed out before, all the nodes in the faulty set located by the precise diagnosis strategy are faulty; while the faulty set located by the pessimistic diagnosis strategy may contain at most one fault-free node. To increase the degree of diagnosability of a multiprocessor system, Somani et al. proposed the t/k-diagnosis strategy [19–21]. Under this strategy, the tested faulty set may contain at most k fault-free nodes ($k \geqslant 0$). They proved that the n-dimensional hypercube is t/k-diagnosable ($n \geqslant 4$ and $0 \leqslant k \leqslant n$), where $t = (k + 1)n - \frac{1}{2}(k + 1)(k + 2) + 1$. Furthermore, it was proved that the n-dimensional BC graphs, including the n-dimensional hypercube, crossed cube and Möbius cube, are t'/k-diagnosable ($n \geqslant 4$ and $0 \leqslant k \leqslant n$), where $t' = (k + 1)n - \frac{1}{2}(k + 1)(k + 2) + 1$ [10]. We can easily find that $t' = n$ when $k = 0$ and $t' = 2n - 2$ when $k = 1$, which match, respectively, the diagnosability of all the n-dimensional BC graphs under the precise and pessimistic diagnosis strategies. It is clear that t' increases by $n - k$ as the number of fault-free nodes included in the fault-set increases from $k - 1$ to k for $0 \leqslant k \leqslant n - 1$. t' is maximal for $k = n - 2$ and $k = n - 1$.

The diagnosability of DCC linear congruential graphs is similar to that of BC graphs under the precise and pessimistic diagnosis strategies. That is, each DCC linear congruential graph is t_p-diagnosable and t_1/t_1-diagnosable, respectively, under the two diagnosis strategies, where t_p is its degree and t_1 doubles its degree minus a constant. Then, one question is about the diagnosability of DCC linear congruential graphs under the t/k-diagnosis strategy. One may ask whether similar results apply to BC graphs. This is an interesting question worth exploring in the future research.

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