J. Parallel Distrib. Comput. 72 (2012) 58–69

Contents lists available at SciVerse ScienceDirect

# J. Parallel Distrib. Comput.

journal homepage: www.elsevier.com/locate/jpdc

# Independent spanning trees on twisted cubes

Yan Wang [a,b], Jianxi Fan [a,*], Guodong Zhou [a], Xiaohua Jia [c]

[a] School of Computer Science and Technology, Soochow University, Suzhou 215006, China
[b] Provincial Key Laboratory for Computer Information Processing Technology, Soochow University, China
[c] Department of Computer Science, City University of Hong Kong, Hong Kong

ARTICLE INFO

*Article history:*
Received 21 March 2011
Received in revised form
8 August 2011
Accepted 3 September 2011
Available online 14 September 2011

*Keywords:*
Broadcasting
Independent spanning tree
Fault tolerance
Twisted cube

ABSTRACT

Multiple independent spanning trees have applications to fault tolerance and data broadcasting in distributed networks. There are two versions of the $n$ independent spanning trees conjecture. The vertex (edge) conjecture is that any $n$-connected ($n$-edge-connected) graph has $n$ vertex-independent spanning trees (edge-independent spanning trees) rooted at an arbitrary vertex. Note that the vertex conjecture implies the edge conjecture. The vertex and edge conjectures have been confirmed only for $n$-connected graphs with $n \leq 4$, and they are still open for arbitrary $n$-connected graph when $n \geq 5$. In this paper, we confirm the vertex conjecture (and hence also the edge conjecture) for the $n$-dimensional twisted cube $TQ_n$ by providing an $O(N \log N)$ algorithm to construct $n$ vertex-independent spanning trees rooted at any vertex, where $N$ denotes the number of vertices in $TQ_n$. Moreover, all independent spanning trees rooted at an arbitrary vertex constructed by our construction method are isomorphic and the height of each tree is $n+1$ for any integer $n \geq 2$.



## 1. Introduction

Multiprocessor interconnect networks (in brief, interconnection networks or networks) play critical roles in parallel computing systems. An interconnection network can be represented by a graph, where a vertex is a processor and an edge is a communication link. In this paper, we use graphs and networks interchangeably.

Broadcasting, which is the process of sending a message from the source vertex to all other vertices in a network, has wide applications to the control of distributed systems and to parallel computing. Broadcasting can be accomplished by means of message dissemination in such a way that each vertex repeatedly receives and forwards messages. However, some of the vertices and/or links in networks may be faulty. We say that the broadcasting will be successful if all the healthy vertices in the network finally obtain the correct message from the source vertex within a certain limit of time.

There are two types of faults which are usually supposed in the study of broadcasting in networks. One is the crash type (i.e., a faulty vertex or link does not transmit any message) and the other is the Byzantine type (i.e., a faulty vertex or link may alter arbitrarily the messages that pass through it or even fabricate a false message). It has been shown that broadcasting using independent spanning trees is the most efficient and the most reliable against the crash and Byzantine faults [3,2,23]. If $G$ is an $n$-connected graph and there exist $n$ independent spanning trees rooted at the source vertex on $G$, supposing that no vertices have priori information about faults in a network, fault tolerance can be achieved by sending $n$ copies of the message along different channels (i.e., independent spanning trees rooted at the source vertex), so that a message corrupted at a place can spoil only one channel for each destination. If the source vertex is faultless, then there exists a broadcasting scheme that tolerates up to $n-1$ faults of the crash type and up to $\lfloor (n-1)/2 \rfloor$ faults of the Byzantine type even in the worst case [3].

Let $V(G)$ denote the vertex set and $E(G)$ denote the edge set of $G$. For two vertices $x, y \in V(G)$, let $P$ and $Q$ denote two paths from $x$ to $y$ in $G$. $P$ and $Q$ are *edge-disjoint* if $E(P) \cap E(Q) = \emptyset$. $P$ and $Q$ are *internally vertex-disjoint* if they are edge-disjoint and $V(P) \cap V(Q) = \{x, y\}$. A subgraph $T$ of $G$ is a *spanning tree* of $G$ if $T$ is a tree and $V(T) = V(G)$. Two spanning trees $T$ and $T'$ of $G$ are *vertex-independent* (resp., *edge-independent*) if $T$ and $T'$ are rooted at the same vertex, say $r$, and for each vertex $v$ ($\neq r$) in $G$, the two paths from $r$ to $v$, one path in each tree, are internally vertex-disjoint (resp., edge-disjoint). A set of spanning trees of $G$ are vertex-independent (resp., edge-independent) if they are pairwise vertex-independent (resp., pairwise edge-independent).

The study of independent spanning trees started with Itai and Rodeh [23], where they proposed a multitree approach to reliability in distributed networks. They provided a linear time algorithm that, given any vertex $r$ in a 2-connected graph $G$, finds two independent spanning trees of $G$ rooted at $r$. Later, Cheriyan

* Corresponding author.
*E-mail address:* jxfan@suda.edu.cn (J. Fan).

0743-7315/$ – see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.jpdc.2011.09.002

and Maheshwari [5] proved that for any vertex $r$ in a 3-connected graph $G$, three independent spanning trees of $G$ rooted at $r$ can be found in $O(|V(G)|\,|E(G)|)$ time.

Zehavi and Itai [39] also proved that every 3-connected graph contains three independent spanning trees rooted at any vertex, and they stated two versions of the $n$ independent spanning trees conjecture:

(1) Vertex Conjecture: Any $n$-connected graph has $n$ vertex-independent spanning trees rooted at an arbitrary vertex.
(2) Edge Conjecture: Any $n$-edge-connected graph has $n$ edge-independent spanning trees rooted at an arbitrary vertex.

Zehavi and Itai raised the question: It would be interesting to show that either the vertex conjecture implies the edge conjecture, or vice versa. Khuller and Schieber [25] proved that the vertex conjecture implies the edge conjecture. The vertex and edge conjectures have been confirmed only for $k$-connected graphs with $k \le 4$ in [5,6,23,24], and they are still open for arbitrary $k$-connected graphs when $k \ge 5$. However, by providing the construction schemes of independent spanning trees, the conjectures have been proved to hold for several restricted classes of graphs such as planar graphs [21–23,30,31], product graphs [32], hypercubes [33,34], locally twisted cubes [16,28,29], star graphs [14], and so forth. Note that the development of algorithms for constructing independent spanning trees tends toward pursuing two research goals, one is the design of efficient construction schemes [30,31] and the other is reducing the heights of independent spanning trees [15,23,38].

In general it is very hard to construct $n$ independent spanning trees rooted at an arbitrary vertex of a given $n$-connected graph. In this paper we focus on the construction of independent spanning trees on twisted cubes.

The twisted cube [1] is an attractive variant of the hypercube. The diameter, wide diameter, and faulty diameter of $n$-dimensional twisted cube, denoted by $\mathrm{TQ}_n$, are about half of those of the $n$-dimensional hypercube [1,4]. It was proved that $\mathrm{TQ}_n$ is $(n-2)$-fault-tolerant Hamiltonian, $(n-3)$-fault-tolerant Hamiltonian-connected [20]. In particular, a stronger result about Hamiltonian property can be seen in [36]. And paths [9,13], cycles [10], 2-dimensional meshes [26], multi-dimensional tori [27] can be embedded into twisted cube, respectively. According to [12], $\mathrm{TQ}_n$ is a bijective connection graph and thus has the same diagnosability as the $n$-dimensional hypercube under the $t/k$-diagnosis strategy based on the well-known PMC diagnostic model. Recently, in bijective connection networks, Fan et al. proposed efficient fault-tolerant routing algorithms with restricted faulty edge or node set [8,11], which are suitable for twisted cubes. There are also other variants of the hypercube, e.g., the crossed cube, locally twisted cube. The topological properties of hypercube and its variants have been discussed [7,17–19,37].

In [35], Yang proposed an $O(N \log N)$ algorithm to construct $n$ edge-disjoint spanning trees on $\mathrm{TQ}_n$, where $N$ is the number of vertices in $\mathrm{TQ}_n$. However, Yang did not solve the vertex conjecture for $\mathrm{TQ}_n$.

In this paper, we provide an algorithm to generate $n$ vertex-independent spanning trees rooted at an arbitrary vertex on $\mathrm{TQ}_n$. Thus we confirm the vertex (also the edge) conjecture for $\mathrm{TQ}_n$. Unless otherwise specified, in this paper, we use independent spanning trees to denote vertex-independent spanning trees.

The major contributions are as follows.

(1) We improve the definition of twisted cubes, which extends the dimensions of twisted cubes to all the positive integers.
(2) We prove that there are $n$ independent spanning trees rooted at any vertex on $\mathrm{TQ}_n$ and present a recursive construction method.
(3) We prove that all independent spanning tree rooted at an arbitrary vertex constructed by our construction method are isomorphic and the height of each tree is $n+1$ for any integer $n \ge 2$.
(4) We give an $O(N \log N)$ algorithm to construct $n$ independent spanning trees rooted at any vertex on $\mathrm{TQ}_n$, where $N$ denotes the number of vertices in $\mathrm{TQ}_n$.

The rest of this paper is organized as follows: Section 2 provides the preliminaries. Section 3 gives the construction method of $n$ independent spanning trees rooted at any vertex on $\mathrm{TQ}_n$. Section 4 analyzes two properties of independent spanning trees constructed on $\mathrm{TQ}_n$, i.e., the isomorphism and height. Section 5 gives an algorithm to construct $n$ independent spanning trees rooted at any vertex on $\mathrm{TQ}_n$ and analyzes time complexity of the algorithm. In Section 6, we conclude this paper.

## 2. Preliminaries

Before presenting the construction method of independent spanning trees on twisted cubes, we first give some notations and related properties.

Given a simple graph $G$, if $V' \subseteq V(G)$, the subgraph of $G$ induced by the vertex subset $V'$ is denoted by $G[V']$. For a tree $T$ rooted at a vertex $r$, the parent of a vertex $x$ ($\neq r$) in $T$ is denoted by parent $(x, T)$, and we use ancestor $(r, x, T)$ to denote the set of vertices in the path from $r$ to $x$ in $T$. The height of $T$ is the maximum distance of the paths from $r$ to any other vertices in $T$, and we use Height $(T)$ to denote the height of $T$. Meanwhile, the degree of a vertex $y$ in $T$ (denoted by $\deg(y, T)$) is defined as the number of its adjacent edges.

For any integer $n \ge 1$, a binary string $u$ of length $n$ is denoted by $u_{n-1}u_{n-2}\dots u_0$, where $u_i \in \{0, 1\}$ for any integer $i \in \{0, 1, \dots, n-1\}$. The complement of $u_i$ is denoted by $\overline{u_i} = 1 - u_i$. Besides, $u_0$ is called the first bit of $u$ and $u_{n-1}$ is called the last bit of $u$. For any integer $i$ with $0 \le i \le n-1$, we define $\theta(u, i) = u_i \oplus u_{i-1} \oplus \dots \oplus u_0$, where $\oplus$ is the exclusive operation. Let $x = x_{m-1}\dots x_1x_0$ be a binary string of length $m$. We use $xu$ to denote the binary string $x_{m-1}x_{m-2}\dots x_0u_{n-1}u_{n-2}\dots u_0$, whose length is $m+n$. Furthermore, let $U \subset \{0, 1\}^n$. That is, $U$ is a set of some binary strings of length $n$, then we use $xU$ to denote the set $\{xu|u \in U\}$. For any $(v, w) \in E(G)$, we use $x(v, w)$ to denote $(xv, xw)$, i.e., $x(v, w) = (xv, xw)$; and if $P \subseteq E(G)$, then we define $xP = \{x(v, w)|(v, w) \in P\}$.

A graph $G_1$ is isomorphic to another graph $G_2$ (denoted by $G_1 \cong G_2$) if and only if there exists a bijection $f : V(G_1) \to V(G_2)$ such that for any two vertices $u, v \in V(G_1)$, $(u, v) \in E(G_1)$ if and only if $(f(u), f(v)) \in E(G_2)$. We call $f$ the isomorphic mapping function from $G_1$ to $G_2$. For any $(u, v) \in E(G_1)$, we write $f(u, v) = (f(u), f(v))$. Furthermore, we write $f(V(G_1)) = V(G_2)$ and $f(E(G_1)) = E(G_2)$.

The $n$-dimensional twisted cube $\mathrm{TQ}_n$ is an $n$-regular graph of $2^n$ vertices. It can be recursively defined as below [13].

**Definition 1.** The 1-dimensional twisted cube $\mathrm{TQ}_1$ is defined as the complete graph with two vertices labeled 0 and 1. For an odd integer $n \ge 3$, $\mathrm{TQ}_n$ consists of four subcubes $\mathrm{TQ}^{00}_{n-2}$, $\mathrm{TQ}^{01}_{n-2}$, $\mathrm{TQ}^{10}_{n-2}$, and $\mathrm{TQ}^{11}_{n-2}$, where $\mathrm{TQ}^{ab}_{n-2}$ is isomorphic to $\mathrm{TQ}_{n-2}$ and $V(\mathrm{TQ}^{ab}_{n-2}) = \{abx|x \in V(\mathrm{TQ}_{n-2})\}$ and $E(\mathrm{TQ}^{ab}_{n-2}) = \{(abx, aby)|(x, y) \in E(\mathrm{TQ}_{n-2})\}$ for $a, b \in \{0, 1\}$. $V(\mathrm{TQ}_n) = \bigcup_{a,b\in\{0,1\}} V(\mathrm{TQ}^{ab}_{n-2})$ and $E(\mathrm{TQ}_n) = \bigcup_{a,b\in\{0,1\}} E(\mathrm{TQ}^{ab}_{n-2}) \bigcup E'$, where for the vertices $u = u_{n-1}u_{n-2}\dots u_1u_0$, $v = v_{n-1}v_{n-2}\dots v_1v_0 \in V(\mathrm{TQ}_n)$, $(u, v) \in E'$ if $u$ and $v$ satisfy one of the following three conditions:

(1) $u = \overline{v_{n-1}}v_{n-2}\dots v_1v_0$;
(2) $u = \overline{v_{n-1}}\,\overline{v_{n-2}}v_{n-3}\dots v_1v_0$ and $\theta(u, n-3) = 0$;
(3) $u = v_{n-1}\overline{v_{n-2}}v_{n-3}\dots v_1v_0$ and $\theta(u, n-3) = 1$.

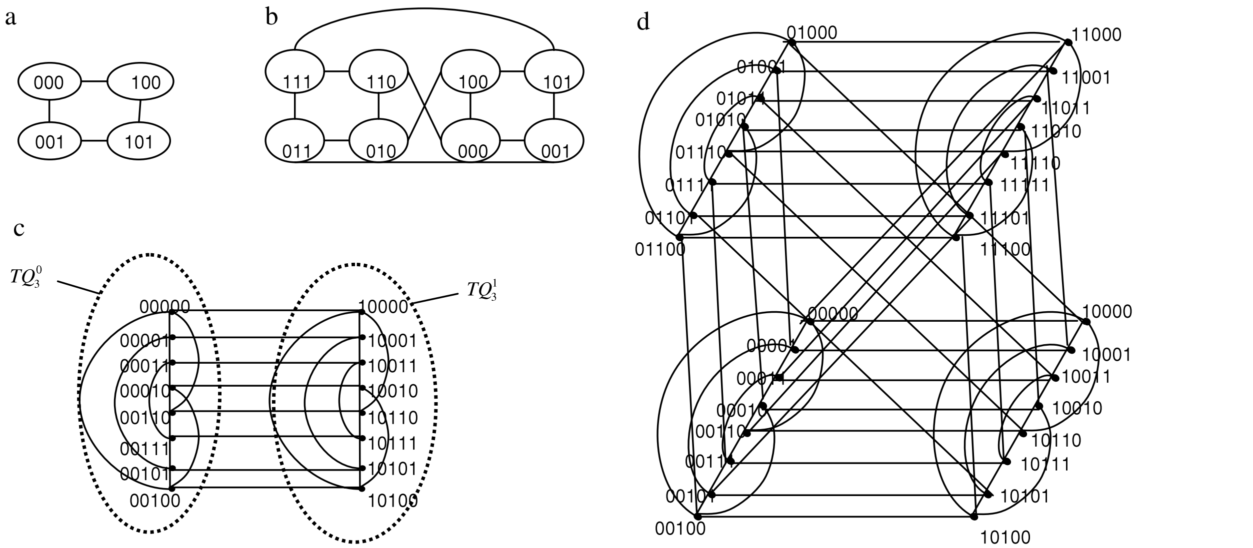



**Fig. 1.** (a) The 2-dimensional twisted cube $TQ_2$. (b) The 3-dimensional twisted cube $TQ_3$. (c) The 4-dimensional twisted cube $TQ_4$. (d) The 5-dimensional twisted cube $TQ_5$.

Fig. 1(b) and (d) demonstrate the 3-dimensional twisted cube $TQ_3$ and the 5-dimensional twisted cube $TQ_5$.

Definition 1 is a recursive definition of $TQ_n$. For convenience of presentation, we present a non-recursive definition of $TQ_n$.

**Definition 2.** For any odd integer $n \geq 1$ and $u = u_{n-1}u_{n-2}\dots u_1u_0 \in V(TQ_n)$, $N_k(u)$ is defined as the $k$-dimensional adjacent vertex of $u$ in $TQ_n$, where $k$ is an integer with $1 \leq k \leq n$.

(1) $N_1(u) = u_{n-1}u_{n-2}\dots u_1\overline{u_0}$;
(2) if $k$ is even, then $N_k(u) = u_{n-1}u_{n-2}\dots\overline{u_k}\dots u_1u_0$;
(3) if $k$ is odd and $\theta(u, k-3) = 0$, then $N_k(u) = u_{n-1}u_{n-2}\dots\overline{u_{k-1}u_{k-2}}\dots u_1u_0$; otherwise, then $N_k(u) = u_{n-1}u_{n-2}\dots u_{k-1}\overline{u_{k-2}}\dots u_1u_0$.

Furthermore, if $V'' \subseteq V(TQ_n)$, then we define $N_k(V'') = \{N_k(u) | u \in V''\}$.

For example, let $u = 10000$ and $v = 10001$. Then $N_1(u) = 10001$, $N_2(u) = 10100$, $N_3(u) = 10110$, $N_4(u) = 00000$, $N_5(u) = 01000$; $N_1(v) = 10000$, $N_2(v) = 10101$, $N_3(v) = 10011$, $N_4(v) = 00001$, $N_5(v) = 11001$.

Definition 1 gives the definition of $TQ_n$ for an odd integer $n \geq 1$. However, there is no definition of $TQ_n$ for any even integer $n \geq 2$. In this paper, for convenience of discussion, we extend the recursive definition of $TQ_n$ to any integer $n \geq 1$ as follows.

**Definition 3.** The 1-dimensional twisted cube $TQ_1$ is defined as the complete graph with two vertices labeled 0 and 1. For any integer $n \geq 1$,

(1) if $n$ is even, the $n$-dimensional twisted cube is divided into two types: 0-type $n$-dimensional twisted cube 0-$TQ_n$ and 1-type $n$-dimensional twisted cube 1-$TQ_n$. For any integer $k \in \{0, 1\}$, $V(k\text{–}TQ_n) = \{ikx | i \in \{0, 1\}, x \in V(TQ_{n-1})\}$ and $E(k\text{–}TQ_n) = \bigcup_{i\in\{0,1\}} ikE(TQ_{n-1}) \cup E'$, where for any two vertices $u = u_{n-1}ku_{n-2}\dots u_1u_0$ and $v = v_{n-1}kv_{n-2}\dots v_1v_0 \in V(k\text{–}TQ_n)$, $(u, v) \in E'$ if $u = \overline{v_{n-1}}kv_{n-1}\dots v_1v_0$. For convenience of presentation, we write $jkTQ_{n-1}$ as $TQ_{n-1}^j$ for $j \in \{0, 1\}$;
(2) if $n$ is odd, the definition of $TQ_n$ is equivalent to that of $TQ_n$ in Definition 1. For convenience of presentation, we write $k$–$TQ_{n-1}$ as $TQ_{n-1}^k$.

Whether $n$ is odd or even, we always denote the $n$-dimensional twisted cube as $TQ_n$ in this paper. Fig. 1(a) and (c) demonstrate the 2-dimensional twisted cube $TQ_2$ and the 4-dimensional twisted cube $TQ_4$.

Based on Definitions 1 and 3, we can easily prove the following lemma.

**Lemma 4.** *For any integers n and i with $n \geq 1$ and $i \in \{0, 1\}$, $TQ_n \cong TQ_n^i$.*

Furthermore, based on the definitions of independent spanning trees and ancestor set, we have the following lemma.

**Lemma 5.** *Let $T$ and $T'$ be two spanning trees rooted at vertex $r$ in $G$. $T$ and $T'$ are independent if and only if for every vertex $x$ in $G$, $x \neq r$, $\{r, x\} \subset \text{ancestor}(r, x, T) \cup \text{ancestor}(r, x, T')$ and $\text{ancestor}(r, x, T) \cap \text{ancestor}(r, x, T') = \{r, x\}$.*

**Notation 6.** *For any integers n and k with $n \geq 1$, $1 \leq k \leq n$ and $v \in V(TQ_n)$, supposing that there are n independent spanning trees $T_1, T_2, \dots,$ and $T_n$ rooted at vertex $v$ on $TQ_n$, if the root $v$ takes $N_k(v)$ as its child in $T_i$, where i is an integer with $1 \leq i \leq n$, we use $T_n^k$ to denote $T_i$.*

Unless otherwise specified, we use $T_n^1, T_n^2, \dots,$ and $T_n^n$ to denote $n$ independent spanning trees on $TQ_n$ in this paper. And for convenience of explanation, we may represent a vertex in independent spanning trees by a decimal value. For example, four independent spanning trees rooted at 0 on $TQ_4$ shown in Fig. 2 are denoted by $T_4^1$, $T_4^2$, $T_4^3$ and $T_4^4$ from left to right, respectively.

## 3. Construction of independent spanning trees on $TQ_n$

In this section, we will discuss the construction of $n$ independent spanning trees on $TQ_n$, where a recursive construction method which can generate $n$ independent spanning trees rooted at any vertex on $TQ_n$ from $n-1$ independent spanning trees on $TQ_{n-1}$ is presented.

For any vertex $v \in V(TQ_{n-1})$, suppose that $n-1$ independent spanning trees $T_{n-1}^1, T_{n-1}^2, \dots,$ and $T_{n-1}^{n-1}$ rooted at $v$, where the child vertex of the root $v$ in $T_{n-1}^j (j \in \{1, 2, \dots, n-1\})$ is $N_j(v)$, have been constructed on $TQ_{n-1}$. According to Lemma 4, we can obtain an isomorphic mapping function $\varphi$ from $TQ_{n-1}$ to $TQ_{n-1}^0$.

**Notation 7.** *For any integers n and i with $n \geq 2$ and $1 \leq i \leq n-1$, we use $T_{n-1}^{0,i}$ to denote $(\varphi(V(T_{n-1}^i)), \varphi(E(T_{n-1}^i)))$. That is, $T_{n-1}^{0,i} = (\varphi(V(T_{n-1}^i)), \varphi(E(T_{n-1}^i)))$.*

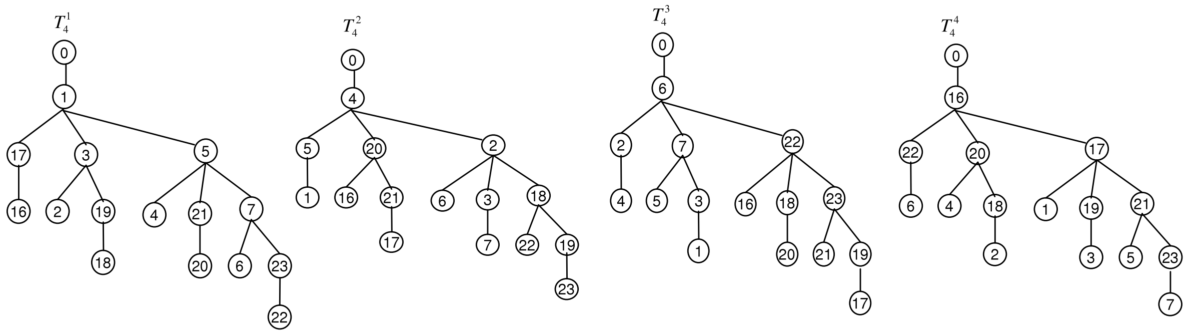



**Fig. 2.** Four independent spanning trees rooted at 0 on $TQ_4$ constructed by the proof of Theorem 17.

By Notation 7, we can easily prove the following lemma.

**Lemma 8.** *$T_{n-1}^{0,1}, T_{n-1}^{0,2}, \ldots,$ and $T_{n-1}^{0,n-1}$ are $n-1$ independent spanning trees rooted at $v$ on $\mathrm{TQ}_{n-1}^0$.*

In the following, we will construct $n-1$ independent spanning trees on $\mathrm{TQ}_{n-1}^1$ by using $T_{n-1}^{0,1}, T_{n-1}^{0,2}, \ldots,$ and $T_{n-1}^{0,n-1}$ on $\mathrm{TQ}_{n-1}^0$. We first give the following constructive rule.

**Rule 9.** *For any integers n and k with $n \geq 2$ and $1 \leq k \leq n-1$,*

1. *if n is even, by changing the last bit of each vertex in $T_{n-1}^{0,k}$ into its complement, we obtain one graph, denoted by $T_{n-1}^{1,k}$;*
2. *if n is odd,* (i) *if $\theta(v, n-3) = 0$, by changing the $(n-1)$-th bit and $(n-2)$-th bit of each vertex in $T_{n-1}^{0,k}$ into their complements, we obtain one graph, denoted by $T_{n-1}^{1,k}$;* (ii) *if $\theta(v, n-3) = 1$, by changing the $(n-2)$-th bit of each vertex in $T_{n-1}^{0,k}$ into its complement, we obtain one graph, denoted by $T_{n-1}^{1,k}$.*

From Rule 9 and Definition 2, we can easily verify the following result.

**Lemma 10.** *$T_{n-1}^{1,1}, T_{n-1}^{1,2}, \ldots,$ and $T_{n-1}^{1,n-1}$ are $n-1$ independent spanning trees rooted at $N_n(v)$ on $\mathrm{TQ}_{n-1}^1$. Furthermore, for any integer k with $1 \leq k \leq n-1$, $T_{n-1}^{0,k} \cong T_{n-1}^{1,k}$.*

**Definition 11.** For any vertex $u \in V(T_{n-1}^{0,k})$ $(1 \leq k \leq n-1)$, we define the corresponding vertex $(\mathrm{CP}(u))$ of $u$ in $T_{n-1}^{1,k}$ as follows:

(1) for any even integer $n \geq 2$, let $u = u_{n-1}0u_{n-2}\ldots u_1u_0 \in V(T_{n-1}^{0,k})$. Then $\mathrm{CP}(u) = \overline{u_{n-1}}0u_{n-2}\ldots u_1u_0$;

(2) for any odd integer $n \geq 3$, let $u = u_{n-1}u_{n-2}\ldots u_1u_0 \in V(T_{n-1}^{0,k})$. (i) If $\theta(v, n-3) = 0$, then $\mathrm{CP}(u) = \overline{u_{n-1}u_{n-2}}\ldots u_1u_0$; (ii) if $\theta(v, n-3) = 1$, then $\mathrm{CP}(u) = u_{n-1}\overline{u_{n-2}}\ldots u_1u_0$.
If $V' \subset V(T_{n-1}^{0,k})$, we use $\mathrm{CP}(V') = \{\mathrm{CP}(u) | u \in V'\}$ to denote the corresponding vertex set of $V'$ in $T_{n-1}^{1,k}$. Let $G'$ be a subgraph of $T_{n-1}^{0,k}$, then $\mathrm{CP}(G')$ is defined as the corresponding subgraph of $G'$ in $T_{n-1}^{1,k}$, where $V(\mathrm{CP}(G')) = \{\mathrm{CP}(u) | u \in V(G')\}$ and $E(\mathrm{CP}(G')) = \{(\mathrm{CP}(u), \mathrm{CP}(v)) | (u, v) \in E(G')\}$.

By Lemma 10 and Definition 11, the following lemma is obvious and its proof is omitted.

**Lemma 12.** $\mathrm{CP}(G') \cong G'$.

From above discussion, we have constructed $n-1$ independent spanning trees on $\mathrm{TQ}_{n-1}^1$ by using $n-1$ independent spanning trees on $\mathrm{TQ}_{n-1}^0$. In the following, we will give the definition of partition vertex. For any integer $j$ with $1 \leq j \leq n-1$, by connecting each partition vertex in $T_{n-1}^{0,j}$ with its $n$-dimensional adjacent vertex in $T_{n-1}^{1,j}$, we can construct the $n$ independent spanning trees on $\mathrm{TQ}_n$.

**Definition 13.** For any integers $n$ and $k$ with $n \geq 2$, $1 \leq k \leq n-1$ and $v \in V(\mathrm{TQ}_{n-1}^0)$, if $T_{n-1}^{0,k}$ is an independent spanning tree rooted at $v$ on $\mathrm{TQ}_{n-1}^0$ and the child vertex of $v$ is $N_k(v)$, then we recursively define a partition vertex set $(\mathrm{ST}_{n-1}^k)$ in $T_{n-1}^{0,k}$ as follows:

(1) for $n = 2$, $\mathrm{ST}_1^1 = \{N_1(v)\}$;

(2) for any $n \geq 3$, $\mathrm{ST}_{n-1}^i = \mathrm{ST}_{n-2}^i \cup N_{n-1}(\mathrm{ST}_{n-2}^i)$ for any integer $i$ with $1 \leq i \leq n-2$, and $\mathrm{ST}_{n-1}^{n-1} = \{N_{n-1}(v)\}$.
We call a vertex in the partition vertex set $\mathrm{ST}_{n-1}^k$ a partition vertex in $T_{n-1}^{0,k}$.

For example, let the root $v = 0$. By Definition 2, $N_1(v) = 1$. According to Definition 13, $\mathrm{ST}_1^1 = \{N_1(v)\} = \{1\}$. Similarly, $N_2(\mathrm{ST}_1^1) = \{5\}$. Thus $\mathrm{ST}_2^1 = \mathrm{ST}_1^1 \cup N_2(\mathrm{ST}_1^1) = \{1, 5\}$.

Based on Definitions 3, 11 and 13, we can give the following three associated lemmas.

**Lemma 14.** *For any vertex $u \in V(T_{n-1}^{0,k})$, let $w = N_{n-1}(u)$. Then*

(1) *for any even integer $n \geq 2$, $\mathrm{CP}(u) = N_n(u)$.*

(2) *for any odd integer $n \geq 3$, if $\theta(v, n-3) = \theta(u, n-3)$, then $\mathrm{CP}(u) = N_n(u)$; otherwise, $\mathrm{CP}(w) = N_n(u)$ and $\mathrm{CP}(u) = N_n(w)$.*

**Proof.** (1) For any even integer $n \geq 2$, let $u = u_{n-1}0u_{n-2}\ldots u_1u_0 \in V(T_{n-1}^{0,k})$. By Definition 11, $\mathrm{CP}(u) = \overline{u_{n-1}}0u_{n-2}\ldots u_1u_0$; and by Definition 2, $N_n(u) = \overline{u_{n-1}}0u_{n-2}\ldots u_1u_0$. Thus $\mathrm{CP}(u) = N_n(u)$.

(2) For any odd integer $n \geq 3$, let $u = u_{n-1}u_{n-2}\ldots u_1u_0 \in V(T_{n-1}^{0,k})$. We deal with the following cases.

Case 1. $\theta(v, n-3) = 0$. By Definition 11, $\mathrm{CP}(u) = \overline{u_{n-1}u_{n-2}}\ldots u_1u_0$. We have the following sub-cases.

Case 1.1. $\theta(u, n-3) = 0$. By Definition 2, $N_n(u) = \overline{u_{n-1}u_{n-2}}\ldots u_1u_0$. Therefore, $\mathrm{CP}(u) = N_n(u)$.

Case 1.2. $\theta(u, n-3) = 1$. By Definition 2, $N_n(u) = u_{n-1}\overline{u_{n-2}}\ldots u_1u_0$. Since $w = N_{n-1}(u) = \overline{u_{n-1}}u_{n-2}\ldots u_1u_0$, then by Definitions 11 and 2, $\mathrm{CP}(w) = u_{n-1}\overline{u_{n-2}}\ldots u_1u_0$ and $N_n(w) = \overline{u_{n-1}u_{n-2}}\ldots u_1u_0$. Therefore, $\mathrm{CP}(w) = N_n(u)$ and $\mathrm{CP}(u) = N_n(w)$.

Case 2. $\theta(v, n-3) = 1$. By Definition 11, $\mathrm{CP}(u) = u_{n-1}\overline{u_{n-2}}\ldots u_1u_0$. We have the following sub-cases.

Case 2.1. $\theta(u, n-3) = 1$. By Definition 2, $N_n(u) = u_{n-1}\overline{u_{n-2}}\ldots u_1u_0$. Therefore, $\mathrm{CP}(u) = N_n(u)$.

Case 2.2. $\theta(u, n-3) = 0$. By Definition 2, $N_n(u) = \overline{u_{n-1}u_{n-2}}\ldots u_1u_0$. Since $w = N_{n-1}(u) = \overline{u_{n-1}}u_{n-2}\ldots u_1u_0$, by Definitions 2 and 11, $\mathrm{CP}(w) = \overline{u_{n-1}u_{n-2}}\ldots u_1u_0$ and $N_n(w) = u_{n-1}\overline{u_{n-2}}\ldots u_1u_0$. Therefore, $\mathrm{CP}(w) = N_n(u)$ and $\mathrm{CP}(u) = N_n(w)$.

In summary, the lemma holds. □

According to Definition 13, for any integers $n$ and $i$ with $n \geq 3$ and $1 \leq i \leq n-2$, $\mathrm{ST}_{n-1}^i = \mathrm{ST}_{n-2}^i \cup N_{n-1}(\mathrm{ST}_{n-2}^i)$. Thus for any $u \in \mathrm{ST}_{n-2}^i$, there exists a vertex $w \in N_{n-1}(\mathrm{ST}_{n-2}^i)$ such that $w = N_{n-1}(u)$. Moreover, by Lemma 14, $\{\mathrm{CP}(u), \mathrm{CP}(w)\} = \{N_n(u), N_n(w)\}$. Therefore, we can easily verify the following lemma.

**Lemma 15.** $N_n(\mathrm{ST}_{n-1}^i) = \mathrm{CP}(\mathrm{ST}_{n-1}^i)$.

**Lemma 16.** *For any integers $n, i$ and $j$ with $n \geq 2, 1 \leq i, j \leq n-1, i \neq j$ and $v \in V(\mathrm{TQ}^0_{n-1})$, $\mathrm{ST}^i_{n-1}$ is a partition vertex set in $T^{0,i}_{n-1}$ rooted at $v$. Then (1) $\bigcup_{i=1}^{n-1} \mathrm{ST}^i_{n-1} = V(T^{0,i}_{n-1}) - \{v\}$; (2) $\mathrm{ST}^i_{n-1} \cap \mathrm{ST}^j_{n-1} = \emptyset$.*

**Proof.** We will prove the lemma by induction on $n$.

The lemma clearly holds for $n = 2$. Supposing that the lemma holds for $n = k - 1$ ($k \geq 3$), we consider $n = k$.

By the induction hypothesis, $\bigcup_{i=1}^{k-2} \mathrm{ST}^i_{k-2} = V(T^{0,i}_{k-2}) - \{v\}$ and $\mathrm{ST}^i_{k-2} \cap \mathrm{ST}^j_{k-2} = \emptyset$. By Lemma 10, $T^{0,i}_{k-2} \cong T^{1,i}_{k-2}$, thus $\bigcup_{i=1}^{k-2} \mathrm{CP}(\mathrm{ST}^i_{k-2}) = V(T^{1,i}_{k-2}) - \{N_{k-1}(v)\}$ and $\mathrm{CP}(\mathrm{ST}^i_{k-2}) \cap \mathrm{CP}(\mathrm{ST}^j_{k-2}) = \emptyset$. Meanwhile, by Lemma 15, $N_{k-1}(\mathrm{ST}^i_{k-2}) = \mathrm{CP}(\mathrm{ST}^i_{k-2})$. Thus $\bigcup_{i=1}^{k-2} N_{k-1}(\mathrm{ST}^i_{k-2}) = V(T^{1,k}_{k-2}) - \{N_{k-1}(v)\}$ and $N_{k-1}(\mathrm{ST}^i_{k-2}) \cap N_{k-1}(\mathrm{ST}^j_{k-2}) = \emptyset$. Furthermore, $(\bigcup_{i=1}^{k-2} \mathrm{ST}^i_{k-2}) \cup (\bigcup_{i=1}^{k-2} N_{k-1}(\mathrm{ST}^i_{k-2})) = (V(T^{0,i}_{k-2}) - \{v\}) \cup (V(T^{1,i}_{k-2}) - \{N_{k-1}(v)\}) = V(T^{0,i}_{k-1}) - \{v, N_{k-1}(v)\}$. By Definition 13, $\mathrm{ST}^i_{k-1} = \mathrm{ST}^i_{k-2} \cup N_{k-1}(\mathrm{ST}^i_{k-2})$ and $\mathrm{ST}^{k-1}_{k-1} = \{N_{k-1}(v)\}$. As a result, $\bigcup_{i=1}^{k-1} \mathrm{ST}^i_{k-1} = (\bigcup_{i=1}^{k-2} \mathrm{ST}^i_{k-2}) \cup (\bigcup_{i=1}^{k-2} N_{k-1}(\mathrm{ST}^i_{k-2})) \cup \mathrm{ST}^{k-1}_{k-1} = V(T^{0,i}_{k-1}) - \{v\}$. Since $\mathrm{ST}^i_{k-2} \cap N_{k-1}(\mathrm{ST}^i_{k-2}) = \emptyset$, $\mathrm{ST}^i_{k-1} \cap \mathrm{ST}^j_{k-1} = (\mathrm{ST}^i_{k-2} \cup N_{k-1}(\mathrm{ST}^i_{k-2})) \cap (\mathrm{ST}^j_{k-2} \cup N_{k-1}(\mathrm{ST}^j_{k-2})) = (\mathrm{ST}^i_{k-2} \cap \mathrm{ST}^j_{k-2}) \cup (N_{k-1}(\mathrm{ST}^i_{k-2}) \cap N_{k-1}(\mathrm{ST}^j_{k-2})) = \emptyset$.

In summary, the lemma holds for $n = k$. □

**Theorem 17.** *For any integer $n \geq 1$ and $v \in V(\mathrm{TQ}_n)$, there exist $n$ independent spanning trees rooted at $v$ on $\mathrm{TQ}_n$.*

**Proof.** We will prove the theorem by induction on $n$.

(1) For $n = 1$, there is one independent spanning tree $T^1_1$ rooted at $v$ on $\mathrm{TQ}_1$, where $V(T^1_1) = \{v, N_1(v)\}$ and $E(T^1_1) = \{(v, N_1(v))\}$. Therefore, the theorem clearly holds for $n = 1$.

(2) Supposing that the theorem holds for $n = k - 1$ ($k \geq 2$), we consider $n = k$.

In the following, we first construct $T^1_k, T^2_k, \ldots$, and $T^k_k$ rooted at $v$ on $\mathrm{TQ}_k$, then we prove $T^1_k, T^2_k, \ldots$, and $T^k_k$ are independent spanning trees on $\mathrm{TQ}_k$.

By the induction hypothesis, there exist $k - 1$ independent spanning trees $T^1_{k-1}, T^2_{k-1}, \ldots$, and $T^{k-1}_{k-1}$ rooted at $v$ on $\mathrm{TQ}_{k-1}$. By Lemmas 4 and 8, we can obtain $k - 1$ independent spanning trees $T^{0,1}_{k-1}, T^{0,2}_{k-1}, \ldots$, and $T^{0,k-1}_{k-1}$ rooted at $v$ on $\mathrm{TQ}^0_{k-1}$. According to Rule 9, we can construct $k - 1$ independent spanning trees $T^{1,1}_{k-1}, T^{1,2}_{k-1}, \ldots$, and $T^{1,k-1}_{k-1}$ rooted at $N_k(v)$ on $\mathrm{TQ}^1_{k-1}$. Furthermore, by Definition 13, $\mathrm{ST}^1_1 = \{N_1(v)\}$ and for any integer $i$ with $1 \leq i \leq k-1$, $\mathrm{ST}^i_k = \mathrm{ST}^i_{k-1} \cup N_k(\mathrm{ST}^i_{k-1})$ and $\mathrm{ST}^k_k = N_k(v)$.

For $v \in V(\mathrm{TQ}_k)$, we have the following cases.

Case 1. $v \in V(\mathrm{TQ}^0_{k-1})$. The construction of $T^1_k, T^2_k, \ldots$, and $T^k_k$ can be described as follows.

Step 1: For any vertex $u \in V(T^{0,i}_{k-1})$, if $u \in \mathrm{ST}^i_{k-1}$, then parent $(N_k(u), T^i_k) = u$. Let $V(T^i_k) = V(T^{0,i}_{k-1}) \cup V(T^{1,i}_{k-1})$ and $E(T^i_k) = (E(T^{0,i}_{k-1}) \cup E(T^{1,i}_{k-1}) \cup E') - E''$, where $E' = \{(u, N_k(u)) | u \in \mathrm{ST}^i_{k-1}\}$ and $E'' = E(T^{1,i}_{k-1}[N_k(\mathrm{ST}^i_{k-1})])$.

Step 2: We have the following sub-steps to construct $T^k_k$.

Step 2.1: Let $N_k(v)$ be the single child of the root $v$.

Step 2.2: For any vertex $u \in V(T^{1,i}_{k-1})$, if $u \in N_k(\mathrm{ST}^i_{k-1})$, then parent $(u, T^k_k) = \mathrm{parent}(u, T^{1,i}_{k-1})$.

Step 2.3: For any vertex $w \in V(T^{0,i}_{k-1})$, if $w \in \mathrm{ST}^i_{k-1}$, then parent $(w, T^k_k) = N_k(w)$.

Based on above steps, for $v \in V(\mathrm{TQ}^0_{k-1})$, $T^1_k, T^2_k, \ldots$, and $T^k_k$ rooted at $v$ on $\mathrm{TQ}_k$ have been constructed.

Case 2. $v \in V(\mathrm{TQ}^1_{k-1})$. The construction of $T^1_k, T^2_k, \ldots$, and $T^k_k$ can be described as follows.

Step 1: For any vertex $u \in V(T^{0,i}_{k-1})$, if $u \in \mathrm{ST}^i_{k-1}$, then parent $(u, T^i_k) = N_k(u)$. Let $V(T^i_k) = V(T^{0,i}_{k-1}) \cup V(T^{1,i}_{k-1})$ and $E(T^i_k) = (E(T^{0,i}_{k-1}) \cup E(T^{1,i}_{k-1}) \cup E') - E''$, where $E' = \{(u, N_k(u)) | u \in \mathrm{ST}^i_{k-1}\}$ and $E'' = E(T^{0,i}_{k-1}[\mathrm{ST}^i_{k-1}])$.

Step 2: We have the following sub-steps to construct $T^k_k$.

Step 2.1: Let $N_k(v)$ be the single child of the root $v$.

Step 2.2: For any vertex $u \in V(T^{0,i}_{k-1})$, if $u \in \mathrm{ST}^i_{k-1}$, then parent $(u, T^k_k) = \mathrm{parent}(u, T^{0,i}_{k-1})$.

Step 2.3: For any vertex $w \in V(T^{1,i}_{k-1})$, if $w \in N_k(\mathrm{ST}^i_{k-1})$, then parent $(w, T^k_k) = N_k(w)$.

Based on above steps, for $v \in V(\mathrm{TQ}^1_{k-1})$, $T^1_k, T^2_k, \ldots$, and $T^k_k$ rooted at $v$ on $\mathrm{TQ}_k$ have been constructed.

From above discussion, we can verify that whether $v \in V(\mathrm{TQ}^0_{k-1})$ or $v \in V(\mathrm{TQ}^1_{k-1})$, the constructions of $k$ independent spanning trees are similar. For $1 \leq i \leq k-1$ and $u \in \mathrm{ST}^i_{k-1}$, if $v \in V(\mathrm{TQ}^0_{k-1})$, then $\mathrm{parent}(N_k(u), T^i_k) = u$; otherwise, parent $(u, T^i_k) = N_k(u)$. Furthermore, the constructions of $T^k_k$ are also similar. Thus, without loss of generality, in the following, we simply consider $v \in V(\mathrm{TQ}^0_{k-1})$.

Based on the above construction method, obviously, $T^1_k, T^2_k, \ldots$, and $T^k_k$ are spanning trees rooted at $v$ on $\mathrm{TQ}_k$. In the following, we will prove $T^1_k, T^2_k, \ldots$, and $T^k_k$ are independent spanning trees on $\mathrm{TQ}_k$.

Firstly, we prove that $T^1_k, T^2_k, \ldots$, and $T^{k-1}_k$ are independent.

For any integers $i$ and $j$ with $1 \leq i, j \leq k-1$, $i \neq j$ and $u \in V(\mathrm{TQ}_k) - \{v\}$, by Lemma 5, we will prove $\{v, u\} \subset \mathrm{ancestor}(v, u, T^i_k) \cup \mathrm{ancestor}(v, u, T^j_k)$ and $\mathrm{ancestor}(v, u, T^i_k) \cap \mathrm{ancestor}(v, u, T^j_k) = \{v, u\}$.

By Lemma 8, $T^{0,i}_{k-1}$ and $T^{0,j}_{k-1}$ are independent. And by Lemma 10, $T^{1,i}_{k-1}$ and $T^{1,j}_{k-1}$ are independent. For $u \in V(\mathrm{TQ}_k) - \{v\}$, we have the following cases.

Case 1. $u \in V(\mathrm{TQ}^0_{k-1})$. Since $T^{0,i}_{k-1}$ and $T^{0,j}_{k-1}$ are independent, $\{v, u\} \subset \mathrm{ancestor}(v, u, T^{0,i}_{k-1}) \cup \mathrm{ancestor}(v, u, T^{0,j}_{k-1})$ and ancestor $(v, u, T^{0,i}_{k-1}) \cap \mathrm{ancestor}(v, u, T^{0,j}_{k-1}) = \{v, u\}$. By above construction method, $\mathrm{ancestor}(v, u, T^i_k) = \mathrm{ancestor}(v, u, T^{0,i}_{k-1})$ and ancestor $(v, u, T^j_k) = \mathrm{ancestor}\ (v, u, T^{0,j}_{k-1})$. Therefore, $\{v, u\} \subset$ ancestor $(v, u, T^i_k) \cup \mathrm{ancestor}(v, u, T^j_k)$ and $\mathrm{ancestor}(v, u, T^i_k) \cap$ ancestor $(v, u, T^j_k) = \{v, u\}$.

Case 2. $u \in V(\mathrm{TQ}^1_{k-1})$. We divide all vertices in $\mathrm{ancestor}(v, u, T^i_k)$ into two vertex sets $A_i$ and $B_i$, where $A_i \subset V(T^{0,i}_{k-1})$ and $B_i \subset V(T^{1,i}_{k-1})$. Since $V(T^{0,i}_{k-1}) \cap V(T^{1,i}_{k-1}) = \emptyset$, $A_i \cap B_i = \emptyset$. Similarly, we divide all vertices in $\mathrm{ancestor}(v, u, T^j_k)$ into two vertex sets $A_j$ and $B_j$, where $A_j \subset V(T^{0,j}_{k-1})$ and $B_i \subset V(T^{1,j}_{k-1})$. Thus $A_j \cap B_j = \emptyset$. Since $T^{0,i}_{k-1}$ and $T^{0,j}_{k-1}$ are independent, $\{v\} \subset A_i \cup A_j$ and $A_i \cap A_j = \{v\}$. Meanwhile, since $T^{1,i}_{k-1}$ and $T^{1,j}_{k-1}$ are independent, $\{u\} \subset B_i \cup B_j$ and $B_i \cap B_j = \{u\}$. Therefore, $\mathrm{ancestor}(v, u, T^i_k) \cup \mathrm{ancestor}(v, u, T^j_k) = (A_i \cup B_i) \cup (A_j \cup B_j) \supset \{v, u\}$ and $\mathrm{ancestor}(v, u, T^i_k) \cap \mathrm{ancestor}(v, u, T^j_k) = (A_i \cup B_i) \cap (A_j \cup B_j) = \{v, u\}$.

Based on Cases 1, 2 and Lemma 5, we have proven that $T^1_k, T^2_k, \ldots$, and $T^{k-1}_k$ are independent.

Secondly, for any integer $p$ with $1 \leq p \leq k-1$, we will prove $T^p_k$ and $T^k_k$ are independent. That is, $\{v, u\} \subset \mathrm{ancestor}(v, u, T^p_k) \cup \mathrm{ancestor}(v, u, T^k_k)$ and $\mathrm{ancestor}(v, u, T^p_k) \cap \mathrm{ancestor}(v, u, T^k_k) = \{v, u\}$. For $u \in V(\mathrm{TQ}_k) - \{v\}$, we further have the following cases.

Case 1. $u \in V(\mathrm{TQ}^0_{k-1})$. Based on above construction method and Lemma 16, $u$ must be a leaf vertex in $T^k_k$, and all vertices except leaf vertices and root $v$ in $T^k_k$ are vertices in $V(\mathrm{TQ}^1_{k-1})$. That is, for $\mathrm{ancestor}(v, u, T^k_k)$, we have $\{v, u\} \subset V(\mathrm{TQ}^0_{k-1})$ and $\{v, u\} \subset \mathrm{ancestor}(v, u, T^k_k)$ and $(\mathrm{ancestor}(v, u, T^k_k) - \{v, u\}) \subset V(\mathrm{TQ}^1_{k-1})$. Meanwhile, $\mathrm{ancestor}(v, u, T^p_k) \subset V(\mathrm{TQ}^0_{k-1})$. Therefore, $\{v, u\} \subset \mathrm{ancestor}(v, u, T^p_k) \cup \mathrm{ancestor}(v, u, T^k_k)$ and $\mathrm{ancestor}(v, u, T^p_k) \cap \mathrm{ancestor}(v, u, T^k_k) = \{v, u\}$.

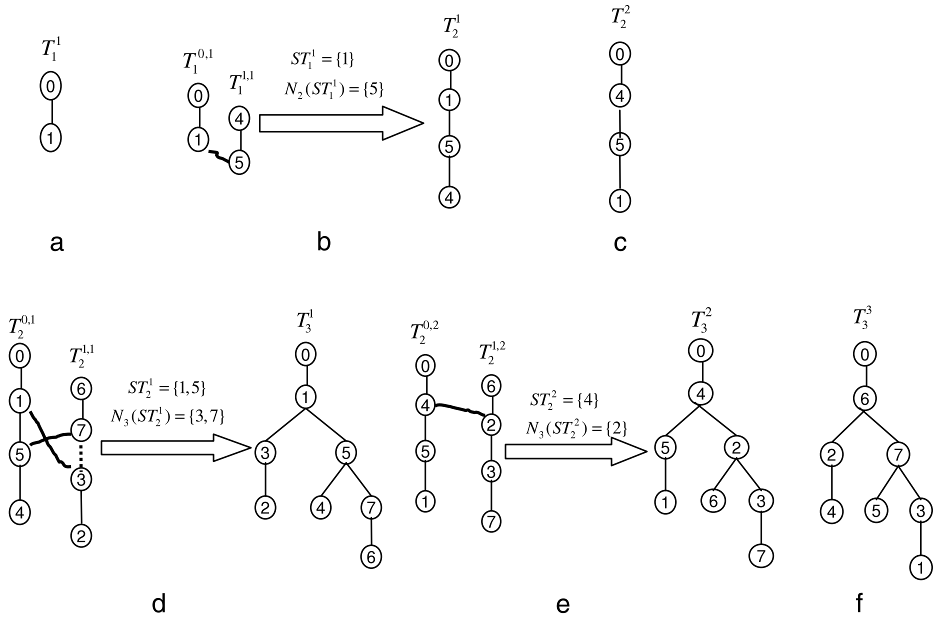



**Fig. 3.** The construction of three independent spanning trees rooted at 0 on $\mathrm{TQ}_3$ according to the proof of Theorem 17.

Case 2. $u \in V(\mathrm{TQ}^1_{k-1})$. We deal with the following sub-cases.

Case 2.1. $u$ is the single child of root $v$ in $T_k^k$. Based on above construction method, $\mathrm{ancestor}(v, u, T_k^k) = \{v, u\}$ and $\{v, u\} \subset \mathrm{ancestor}(v, u, T_k^p)$. Thus, $\{v, u\} \subset \mathrm{ancestor}(v, u, T_k^p) \cup \mathrm{ancestor}(v, u, T_k^k)$ and $\mathrm{ancestor}(v, u, T_k^p) \cap \mathrm{ancestor}(v, u, T_k^k) = \{v, u\}$.

Case 2.2. $u$ is not the single child of root $v$ in $T_k^k$. By above construction method and Lemma 16, there exists an integer $m$ with $1 \le m \le k-1$ such that $u \in N_k(\mathrm{ST}^m_{k-1})$. Based on above construction method, $\mathrm{parent}(u, T_k^m) \in \mathrm{ST}^m_{k-1}$. Thus $(\mathrm{ancestor}(v, u, T_k^m) - \{u\}) \subset V(\mathrm{TQ}^0_{k-1})$. Meanwhile, based on the above constructive method, $\mathrm{parent}(u, T_k^k) \in V(\mathrm{TQ}^1_{k-1})$. Thus, $\{v, u\} \subset \mathrm{ancestor}(v, u, T_k^k)$ and $(\mathrm{ancestor}(v, u, T_k^k) - \{v\}) \subset V(\mathrm{TQ}^1_{k-1})$. As a result, $\{v, u\} \subset \mathrm{ancestor}(v, u, T_k^m) \cup \mathrm{ancestor}(v, u, T_k^k)$ and $\mathrm{ancestor}(v, u, T_k^m) \cap \mathrm{ancestor}(v, u, T_k^k) = \{v, u\}$.

Based on Case 1, Case 2 and Lemma 5, we have proven that $T_k^p (1 \le p \le k-1)$ and $T_k^k$ are independent.

Therefore, $T_k^1, T_k^2, \ldots,$ and $T_k^k$ are $k$ independent spanning trees rooted at any vertex $v$ on $\mathrm{TQ}_k$.

Thus the theorem holds. □

The construction of three independent spanning trees rooted at 0 on $\mathrm{TQ}_3$ is shown in Fig. 3, where $E''$ is shown by dotted lines.

## 4. Two properties of independent spanning trees

In fact, Theorem 17 gives a construction method of $n$ independent spanning trees rooted at any vertex $v$ on $\mathrm{TQ}_n$. In Section 5, we will present an algorithm and analyze its time complexity. In this section, we discuss two properties of independent spanning trees constructed by the proof of Theorem 17, that is, isomorphism and the height of these trees. By the proof of Theorem 17, for $v \in V(\mathrm{TQ}^0_{n-1})$ or $v \in V(\mathrm{TQ}^1_{n-1})$, the construction methods are similar. Without loss of generality, in the following, we simply consider $v \in V(\mathrm{TQ}^0_{n-1})$.

### 4.1. Isomorphism of independent spanning trees

In order to prove that a set of independent spanning trees constructed by the proof of Theorem 17 are isomorphic. We first give the following lemma.

**Lemma 18.** *A set of independent spanning trees constructed on* $\mathrm{TQ}_n$ *by the proof of Theorem 17 satisfy the following properties: for any integers $n$ and $k$ with $n \ge 3$, $1 \le k \le n-1$ and two vertices $u, v \in V(\mathrm{TQ}_n)$, $T_n^k$ is an independent spanning tree rooted at vertex $v$ on $\mathrm{TQ}_n$, where the child of $v$ is $N_k(v)$.*

(1) *If $u$ is the single child of root $v$ in $T_n^n$, then $\deg(u, T_n^n) = n$;*
(2) *if $\deg(u, T_n^k) \ge k+1$, then $u \in \mathrm{ST}_n^k$;*
(3) *if $u \in \mathrm{ST}_n^k$, then $\deg(u, T_n^k) \ge k+1$; and if $u \in N_n(\mathrm{ST}^k_{n-1})$, then $\deg(u, T_n^k) = k+1$;*
(4) *if $\deg(u, T_n^k) = k+1$, then $u = N_n(p)$, where $p = \mathrm{parent}(u, T_n^k)$.*

**Proof.** (1) By the proof of Theorem 17, in $T_n^n$ the single child of root $v$ is $N_n(v)$. Since $N_n(v)$ is the root of $T^{1,k}_{n-1}$, by the proof of Theorem 17, $N_n(v)$ has $n-1$ child vertices in $T_n^n$. Therefore, $\deg(u, T_n^n) = (n-1)+1 = n$.

In the following, we will prove (2), (3) and (4) by induction on $n$. It can be easily verified that the lemma holds for $n = 3$. Supposing that the lemma holds for $n = m-1$ $(m \ge 4)$, we consider $n = m$.

(2) For $1 \le k \le m-1$, we have the following cases.

Case 1. $1 \le k \le m-2$. By the induction hypothesis, in $T^{0,k}_{m-1}$ all the vertices whose degree is greater than or equal to $k+1$ are partition vertices. Meanwhile, by Lemma 10, $T^{0,k}_{m-1} \cong T^{1,k}_{m-1}$. Furthermore, according to Lemma 15, $N_m(\mathrm{ST}^k_{m-1}) = \mathrm{CP}(\mathrm{ST}^k_{m-1})$. Thus each vertex whose degree in $T^{1,k}_{m-1}$ is greater than or equal to $k+1$ is an $m$-dimensional adjacent vertex of a partition vertex in $T^{0,k}_{m-1}$. According to the proof of Theorem 17, $T_m^k$ is constructed by connecting each partition vertex in $T^{0,k}_{m-1}$ with its $m$-dimensional adjacent vertex in $T^{1,k}_{m-1}$, and $V(T_m^k) = V(T^{0,k}_{m-1}) \cup V(T^{1,k}_{m-1})$. Therefore, the degree of the vertices whose degree is less than $k+1$ in $T^{0,k}_{m-1}$ and $T^{1,k}_{m-1}$ remains unchanged in $T_m^k$. As a result, for any $u \in V(T_m^k)$, if $\deg(u, T_m^k) \ge k+1$, then $u \in \mathrm{ST}^k_{m-1}$ or $u \in N_m(\mathrm{ST}^k_{m-1})$. Furthermore, since $\mathrm{ST}_m^k = \mathrm{ST}^k_{m-1} \cup N_m(\mathrm{ST}^k_{m-1})$, $u$ is a partition vertex in $T_m^k$. That is, $u \in \mathrm{ST}_m^k$.

Case 2. $k = m-1$. According to the proof of Theorem 17, in $T^{0,m-1}_{m-1}$, $\mathrm{ST}^{m-1}_{m-1} = N_{m-1}(v)$ and $N_{m-1}(v)$ is the single child of root $v$. By (1), $\deg(N_{m-1}(v), T^{0,m-1}_{m-1}) = m-1$. Meanwhile, by Lemma 10, $T^{0,k}_{m-1} \cong T^{1,k}_{m-1}$. Thus $\deg(\mathrm{CP}(N_{m-1}(v)), T^{1,m-1}_{m-1}) = m-1$. According to Lemma 14, $N_m(N_{m-1}(v)) = \mathrm{CP}(N_{m-1}(v))$. Thus

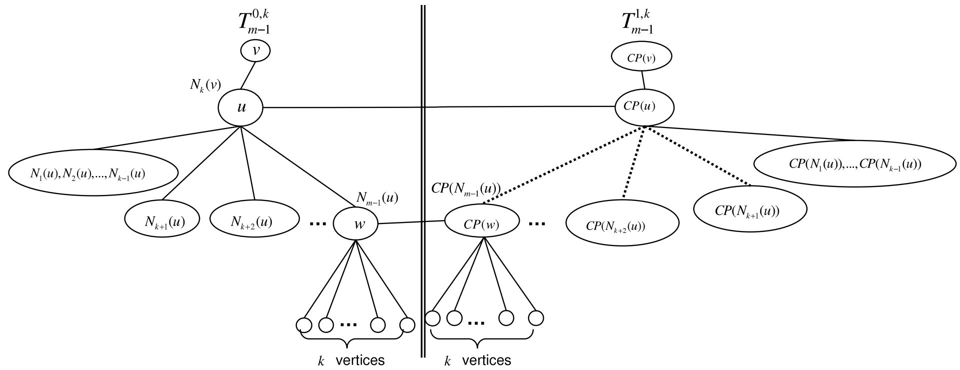



**Fig. 4.** The computation of $\deg(\mathrm{CP}(u), T^k_m)$ where $u \in \mathrm{ST}^k_{m-2}$, $u$ is the single child of root $v$ and parent $(\mathrm{CP}(u), T^k_m) = u$.

$\deg(N_m(N_{m-1}(v)), T^{1,m-1}_{m-1}) = m-1$. Furthermore, by the proof of Theorem 17, $\deg(N_{m-1}(v), T^{m-1}_m) = \deg(N_m(N_{m-1}(v)), T^{m-1}_m) = m$. Thus in $T^{m-1}_m$, the degree of vertices except $N_{m-1}(v)$ and $N_m(N_{m-1}(v))$ is less than $m$, i.e., $k+1$. By the proof of Theorem 17, the degree of the vertices whose degree is less than $k+1$ in $T^{0,m-1}_{m-1}$ and $T^{1,m-1}_{m-1}$ remains unchanged in $T^{m-1}_m$. Thus, for any $u \in V(T^{m-1}_m)$, if $\deg(u, T^{m-1}_m) = k+1$, then $u \in \mathrm{ST}^{m-1}_{m-1}$ or $u \in N_m(\mathrm{ST}^{m-1}_{m-1})$. Furthermore, since $\mathrm{ST}^k_m = \mathrm{ST}^k_{m-1} \cup N_m(\mathrm{ST}^k_{m-1})$, $u$ is a partition vertex in $T^k_m$. That is, $u \in \mathrm{ST}^k_m$.

From Cases 1 and 2, for $1 \le k \le m-1$, if $\deg(u, T^k_m) \ge k+1$, then $u \in \mathrm{ST}^k_m$.

(3) Since $\mathrm{ST}^k_m = \mathrm{ST}^k_{m-1} \cup N_m(\mathrm{ST}^k_{m-1})$, for any $s \in \mathrm{ST}^k_m$, we further have the following cases:

Case 1. $s \in \mathrm{ST}^k_{m-1}$. By the proof of Theorem 17, parent $(N_m(s), T^k_m) = s$. Thus, $\deg(s, T^k_m) = \deg(s, T^{0,k}_{m-1}) + 1$. By the induction hypothesis, for $1 \le k \le m-2$, $\deg(s, T^{0,k}_{m-1}) \ge k+1$. Furthermore, by the proof of Theorem 17, $\mathrm{ST}^{m-1}_{m-1} = N_{m-1}(v)$. By (1), $\deg(s, T^{m-1}_{m-1}) = m-1$. As a result, for $1 \le k \le m-1$, $\deg(s, T^k_m) \ge k+1$.

Case 2. $s \in N_m(\mathrm{ST}^k_{m-1})$. Let $u = \mathrm{parent}(s, T^k_m)$. By the proof of Theorem 17, thus $u \in \mathrm{ST}^k_{m-1}$ and $s = N_m(u)$. Since $\mathrm{ST}^k_{m-1} = \mathrm{ST}^k_{m-2} \cup N_{m-1}(\mathrm{ST}^k_{m-2})$, we have the following cases.

Case 2.1. $u \in \mathrm{ST}^k_{m-2}$. Let $w = N_{m-1}(u)$. Then $w \in N_{m-1}(\mathrm{ST}^k_{m-2})$. By Definition 13, $\mathrm{ST}^k_{m-1} = \mathrm{ST}^k_{m-2} \cup N_{m-1}(\mathrm{ST}^k_{m-2})$. Thus $w \in \mathrm{ST}^k_{m-1}$. By the induction hypothesis, $\deg(w, T^k_{m-1}) = k+1$. By Lemma 14, $N_m(u) = \mathrm{CP}(u)$ or $N_m(u) = \mathrm{CP}(w)$, i.e., $s = \mathrm{CP}(u)$ or $s = \mathrm{CP}(w)$. In the following, we will prove $\deg(s, T^k_m) = k+1$. We deal with the following cases.

Case 2.1.1. $s = \mathrm{CP}(u)$. We further have the following sub-cases.

Case 2.1.1.1. $u$ is the single child of root $v$ in $T^{0,k}_{m-1}$. By Notation 6, $u = N_k(v)$. Meanwhile, by the proof of Theorem 17, $N_k(v)$ is the single child of root $v$ in $T^k_k$. Furthermore, by (1), $\deg(N_k(v), T^k_k) = k$. According to the proof of Theorem 17, among the child vertices of $u$, there are $m-k-1$ partition vertices in $T^{0,k}_{m-1}$, which are $N_{k+1}(u), N_{k+2}(u), \ldots,$ and $N_{m-1}(u)$. Therefore, $\deg(u, T^{0,k}_{m-1}) = k + (m-k-1) = m-1$. And by Lemma 10, $T^{0,k}_{m-1} \cong T^{1,k}_{m-1}$. Therefore, $\deg(\mathrm{CP}(u), T^{1,k}_{m-1}) = m-1$. Then by the proof of Theorem 17, in $T^{1,k}_{m-1}$ the edge set $\{(\mathrm{CP}(u), \mathrm{CP}(N_{k+1}(u))), (\mathrm{CP}(u), \mathrm{CP}(N_{k+2}(u))), \ldots, (\mathrm{CP}(u), \mathrm{CP}(N_{m-1}(u)))\} \subset E''$ ($E''$ is defined in the proof in Theorem 17 and shown by dotted lines in Fig. 4). Therefore, $\deg(s, T^k_m) = \deg(\mathrm{CP}(u), T^k_m) = \deg(\mathrm{CP}(u), T^{1,k}_{m-1}) - (m-k-1) + 1 = k+1$ (see Fig. 4).

Case 2.1.1.2. $u$ is not the single child of the root $v$ in $T^{0,k}_{m-1}$. Then there exists an integer $l$ with $1 \le l \le m-1$ such that $u \in N_l(\mathrm{ST}^k_{l-1})$. By the proof of Theorem 17, among the child vertices of $u$ are $m-l-1$ partition vertices, which are $N_{l+1}(u), N_{l+2}(u), \ldots,$ and $N_{m-1}(u)$. By the induction hypothesis, $\deg(w, T^{0,k}_{m-1}) = k+1$. Thus, $\deg(u, T^{0,k}_{m-1}) = k+1+(m-1)-(l+1)+1 = k+m-l \ge k+1$. Meanwhile, by Lemma 10, $T^{0,k}_{m-1} \cong T^{1,k}_{m-1}$. Therefore, $\deg(\mathrm{CP}(u), T^{1,k}_{m-1}) = k+m-l$. Since there are $m-l-1$ partition vertices among the $k+m-l-1$ child vertices of $u$, then $\deg(s, T^k_m) = \deg(\mathrm{CP}(u), T^k_m) = \deg(\mathrm{CP}(u), T^{1,k}_{m-1}) - (m-l-1) - 1 + 1 = k+1$ (see Fig. 5).

Case 2.1.2. $s = \mathrm{CP}(w)$. By the induction hypothesis, $\deg(w, T^{0,k}_{m-1}) = k+1$. By the proof of Theorem 17, it is impossible for the child vertices of $w$ to be partition vertices in $T^{0,k}_{m-1}$. Therefore, $\deg(s, T^k_m) = \deg(\mathrm{CP}(w), T^k_m) = \deg(\mathrm{CP}(w), T^{1,k}_{m-1}) - 1 + 1 = k+1$.

From Case 2.1.1 and Case 2.1.2, we have the result: $\deg(s, T^k_m) = \deg(\mathrm{CP}(u), T^k_m) = k+1$, where $u \in \mathrm{ST}^k_{m-2}$.

Case 2.2. $u \in N_{m-1}(\mathrm{ST}^k_{m-2})$. Let $p = \mathrm{parent}(u, T^{0,k}_{m-1})$. Then by the proof of Theorem 17, $p \in \mathrm{ST}^k_{m-2}$ and $u = N_{m-1}(p)$. By the induction hypothesis, $\deg(u, T^k_{m-1}) = k+1$. By Lemma 14, $N_m(u) = \mathrm{CP}(u)$ or $N_m(u) = \mathrm{CP}(p)$, i.e., $s = \mathrm{CP}(u)$ or $s = \mathrm{CP}(p)$. In the following, we will prove $\deg(s, T^k_m) = k+1$. We deal with the following cases.

Case 2.2.1. $s = \mathrm{CP}(u)$. Similar to the computation of $\mathrm{CP}(w)$ in Case 2.1.2, thus $\deg(s, T^k_m) = k+1$.

Case 2.2.2. $s = \mathrm{CP}(p)$. Similar to the computation of $\mathrm{CP}(u)$ in Case 2. 1.1, thus $\deg(s, T^k_m) = k+1$.

From Cases 1 and 2, we have proven that if $u \in \mathrm{ST}^k_m$, then $\deg(u, T^k_m) \ge k+1$; and if $u \in N_m(\mathrm{ST}^k_{m-1})$, then $\deg(u, T^k_m) = k+1$.

(4) Let $u$ denote an arbitrary vertex in $T^k_m$ whose degree is greater than or equal to $k+1$. By (2), thus $u \in \mathrm{ST}^k_m$. Since $\mathrm{ST}^k_m = \mathrm{ST}^k_{m-1} \cup N_m(\mathrm{ST}^k_{m-1})$, if $u \in \mathrm{ST}^k_{m-1}$, by the induction hypothesis, $\deg(u, T^{0,k}_{m-1}) \ge k+1$. Furthermore, by the proof of Theorem 17, parent $(N_m(u), T^k_m) = u$. As a result, $\deg(u, T^k_m) = \deg(u, T^{0,k}_{m-1}) + 1 \ge \min\{k+2, m\} \ge k+1$. Therefore, we have $u \in N_m(\mathrm{ST}^k_{m-1})$. Let $p = \mathrm{parent}(u, T^k_m)$. By the proof of Theorem 17, $p \in \mathrm{ST}^k_{m-1}$. According to above discussion, $u = N_m(p)$.

In summary, the lemma holds for $n = m$. □

In the following, in order to prove that the $n$ independent spanning trees constructed in the proof of Theorem 17 are isomorphic, we present the definition of an $n$-dimensional partition tree ($\mathrm{PT}_n$) and some associated lemmas.

**Definition 19.** For any integer $n$ with $n \ge 1$, an $n$-dimensional partition tree $\mathrm{PT}_n$ rooted at $v$ can be recursively defined as follows.

(1) $\mathrm{PT}_1$ rooted at $v$ is a tree with two vertices, where $v$ has a single child.

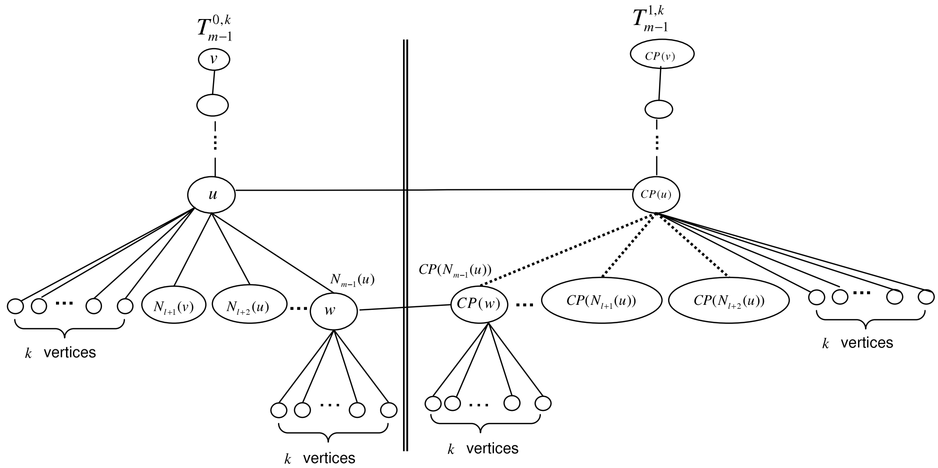



**Fig. 5.** The computation of $\deg(\mathrm{CP}(u), T^k_m)$ where $u \in \mathrm{ST}^k_{m-2}$, $u$ is not the single child of root $v$ and $\mathrm{parent}(\mathrm{CP}(u), T^k_m) = u$.

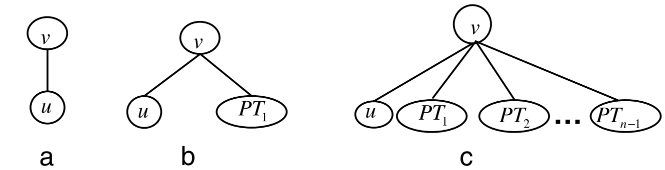



**Fig. 6.** (a) The graph of $\mathrm{PT}_1$. (b) The graph of $\mathrm{PT}_2$. (c) The graph of $\mathrm{PT}_n$.

(2) For any integer $n \geq 2$, an $n$-dimensional partition tree rooted at $v$ is composed of $n$ subtrees $\mathrm{PT}_0, \mathrm{PT}_1, \mathrm{PT}_2, \ldots,$ and $\mathrm{PT}_{n-1}$, where $\mathrm{PT}_0$ is the tree with single vertex and $\mathrm{PT}_i$ ($i = 1, 2, \ldots, n-1$) is an $i$-dimensional partition tree rooted at $v_i$ (see Fig. 6).

By Definition 19, we can easily verify the following lemmas.

**Lemma 20.** $\deg(v, \mathrm{PT}_n) = n$.

**Lemma 21.** *By adding a vertex to each vertex in* $\mathrm{PT}_{n-1}$ *as its child, we can obtain* $\mathrm{PT}_n$.

**Notation 22.** *For any integers n and i with* $n \geq 3$ *and* $1 \leq i \leq n-1$*, we use* $\mathrm{SPT}^i_n$ *to denote* $T^i_n[\mathrm{ST}^i_n]$.

**Lemma 23.** *For any integers n and i with* $n \geq 3$ *and* $1 \leq i \leq n-1$, $\mathrm{SPT}^i_n \cong \mathrm{PT}_{n-i}$.

**Proof.** We will prove the lemma by induction on $n$. We can easily verify that the lemma holds for $n = 3$ (see Fig. 7(a) and (b)). Supposing that the lemma holds for $n = k-1 (k \geq 4)$, we consider $n = k$. By Definition 13, $\mathrm{ST}^i_k = \mathrm{ST}^i_{k-1} \cup N_k(\mathrm{ST}^i_{k-1})$. For any vertex $u \in \mathrm{ST}^i_{k-1}$, by Notation 22, $u \in V(\mathrm{SPT}^i_{k-1})$. By the induction hypothesis, $\mathrm{SPT}^i_{k-1} \cong \mathrm{PT}_{k-1-i}$ (see Fig. 7(c)). Meanwhile, by the proof of Theorem 17, parent $(N_k(u), \mathrm{SPT}^i_k) = u$. By Lemma 21, we can change $\mathrm{PT}_{k-1-i}$ into a $(k-i)$-dimensional partition tree $\mathrm{PT}_{k-i}$. As a result, $\mathrm{SPT}^i_k \cong \mathrm{PT}_{k-i}$ (see Fig. 7(d)).

According to the above discussion, the lemma holds for $n = k$. □

In the following, we will prove that the $n$ independent spanning trees on $\mathrm{TQ}_n$ constructed in the proof of Theorem 17 are isomorphic. Obviously, for $n = 1, 2$, the $n$ independent spanning trees on $\mathrm{TQ}_n$ constructed in the proof of Theorem 17 are isomorphic. In particular, for $n \geq 3$, we have the following theorem.

**Theorem 24.** *For any integers n and i with* $n \geq 3$ *and* $1 \leq i \leq n$*, let* $T^i_n$ *be an independent spanning tree on* $\mathrm{TQ}_n$ *constructed in the proof of Theorem 17. Then* $T^i_n$ *is isomorphic to the tree rooted at u that has v as a child of u and v has* $n-1$ *subtrees* $\mathrm{PT}_1, \mathrm{PT}_2, \ldots,$ *and* $\mathrm{PT}_{n-1}$.

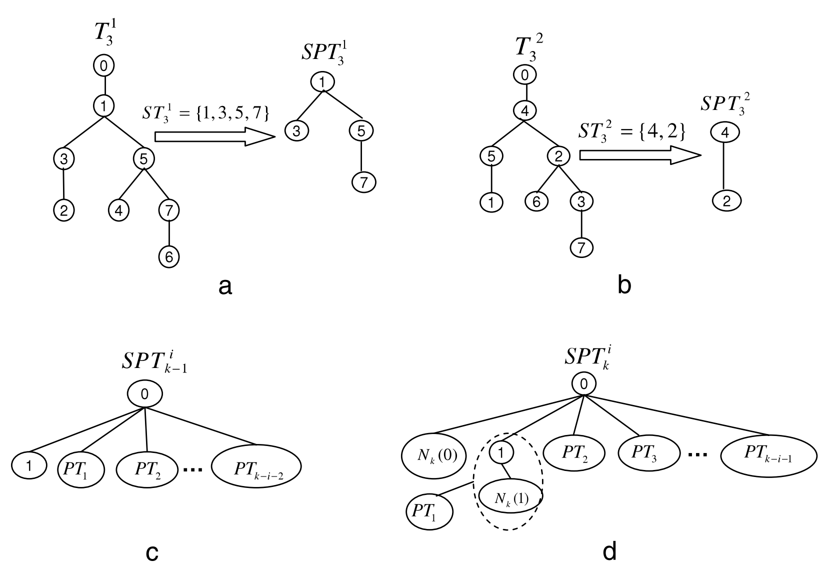



**Fig. 7.** The illustrations of $\mathrm{SPT}^1_3$, $\mathrm{SPT}^2_3$, $\mathrm{SPT}^i_{k-1}$ and $\mathrm{SPT}^i_k$.

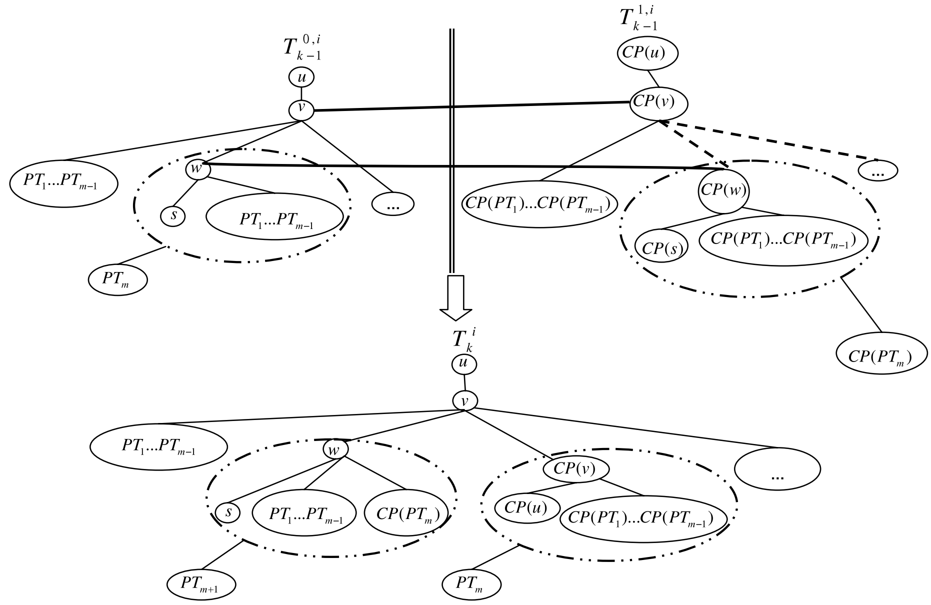



**Fig. 8.** The construction of $T_k^i$ where $1 \le i \le k-1$, $m = i$ and $N_k(v) = \mathrm{CP}(v)$.

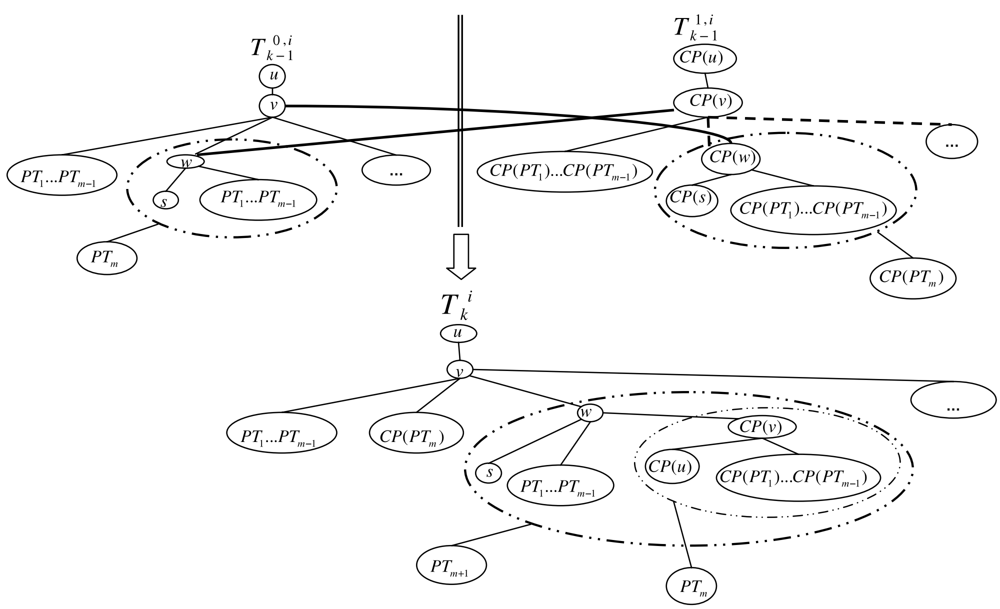



**Fig. 9.** The construction of $T_k^i$ where $1 \le i \le k-1$, $m = i$ and $N_k(v) = \mathrm{CP}(w)$.

**Proof.** We will prove the theorem by induction on $n$.

The theorem clearly holds for $n = 3$. Supposing that the theorem holds for $n = k-1 (k \ge 4)$ (see Fig. 10(a)), we consider $n = k$. We can deal with the following cases.

Case 1. $1 \le i \le k-1$. For any integer $m$ with $1 \le m \le k-2$, let $w \in V(T_{k-1}^{0,i})$ and $w \ne v$ denote the root of $\mathrm{PT}_m$. By Lemma 20, $\deg(w, \mathrm{PT}_m) = m$. Therefore, $\deg(w, T_{k-1}^{0,i}) = m+1$. We further have the following sub-cases.

Case 1.1. $m < i$. Since $\deg(w, T_{k-1}^{0,i}) < i+1$, according to the proof of Theorem 17 and Lemma 18, there is no partition vertex in $\mathrm{PT}_m$. Thus $\mathrm{PT}_m$ remains unchanged in $T_k^i$.

Case 1.2. $m = i$. Since $\deg(w, T_{k-1}^{0,i}) = i+1$, by Lemma 18, $w$ is a partition vertex and $w = N_{k-1}(v)$. By Lemma 14, $N_k(v) = \mathrm{CP}(v)$ or $N_k(v) = \mathrm{CP}(w)$. Hence, we further have the following sub-cases.

Case 1.2.1. $N_k(v) = \mathrm{CP}(v)$. By the proof of Theorem 17, parent $(\mathrm{CP}(v), T_k^i) = v$ and parent $(\mathrm{CP}(w), T_k^i) = w$. Furthermore, by the proof of Theorem 17, in $T_k^i$, $w$ will add $\mathrm{CP}(\mathrm{PT}_m)$ rooted at $\mathrm{CP}(w)$ as its subtree. By Lemma 12, $\mathrm{CP}(\mathrm{PT}_m) \cong \mathrm{PT}_m$. And by Definition 19, $\mathrm{PT}_m$ in $T_{k-1}^{0,i}$ is changed into $\mathrm{PT}_{m+1}$ in $T_k^i$. Meanwhile, by the proof of Theorem 17, in $T_k^i$, $\mathrm{CP}(\mathrm{PT}_1), \mathrm{CP}(\mathrm{PT}_2), \ldots,$ and $\mathrm{CP}(\mathrm{PT}_{m-1})$ in $T_{k-1}^{1,i}$ are changed into $m-1$ subtrees of $\mathrm{CP}(v)$ and the root $\mathrm{CP}(u)$ in $T_{k-1}^{1,i}$ is changed into a child vertex of $\mathrm{CP}(v)$. By Lemma 12, for any integer $l$ with $1 \le l \le m-1$, $\mathrm{CP}(\mathrm{PT}_l) \cong \mathrm{PT}_l$. Thus in $T_k^i$, the vertex $v$ will add $\mathrm{PT}_m$ rooted at $\mathrm{CP}(v)$ as its subtree (see Fig. 8).

Case 1.2.2. $N_k(v) = \mathrm{CP}(w)$. By the proof of Theorem 17, parent $(\mathrm{CP}(v), T_k^i) = w$ and parent $(\mathrm{CP}(w), T_k^i) = v$. By Lemma 12, $\mathrm{CP}(\mathrm{PT}_m) \cong \mathrm{PT}_m$. Thus, $v$ will add $\mathrm{PT}_m$ rooted at $\mathrm{CP}(w)$ as its subtree. Meanwhile, by the proof of Theorem 17, in $T_k^i$, $\mathrm{CP}(\mathrm{PT}_1)$, $\mathrm{CP}(\mathrm{PT}_2), \ldots,$ and $\mathrm{CP}(\mathrm{PT}_{m-1})$ in $T_{k-1}^{1,i}$ are changed into $m-1$

subtrees of $CP(v)$ and the root in $T_{k-1}^{1,i}$ is changed into a child vertex of $CP(v)$. By Lemma 12, for any integer $l$ with $1 \le l \le m-1$, $CP(PT_l) \cong PT_l$. Thus in $T_k^i$, the vertex $w$ will add $PT_m$ rooted at $CP(v)$ as its subtree. By Definition 19, $PT_m$ in $T_{k-1}^{0,i}$ is changed into $PT_{m+1}$ in $T_k^i$ (see Fig. 9).

Case 1.3. $m > i$. By Lemma 18, there are at least two partition vertices in $PT_m$. According to Definition 19, Cases 1.1 and 1.2, in $T_k^i$, the vertex $v$ will add $PT_{m+1}$ as its subtree.

From Cases 1.1–1.3, we can verify the result that for $1 \le i \le k-1$, $T_k^i$ is isomorphic to the tree rooted at $u$ that has $v$ as a child vertex of $u$ and $v$ has $k-1$ subtrees $PT_1, PT_2, \ldots,$ and $PT_{k-1}$ (see Fig. 10(b)).

Case 2. $i = k$. For any integer $j$ with $1 \le j \le k-1$, let $s$ denote any vertex in $V(T_k^k[N_k(ST_{k-1}^j)])$. By the proof of Theorem 17, $parent(s, T_k^k) = parent(s, T_{k-1}^{1,j})$. Meanwhile, by Lemma 10, $T_{k-1}^{0,j} \cong T_{k-1}^{1,j}$. Therefore, $T_k^k[N_k(ST_{k-1}^j)] \cong SPT_{k-1}^j$. Then by Lemma 23, $T_k^k[N_k(ST_{k-1}^j)] \cong SPT_{k-1}^j \cong PT_{k-1-j}$. And according to the proof of Theorem 17, for each vertex in $T_k^k[N_k(ST_{k-1}^j)]$, by adding a vertex as its child, we can obtain $T_k^k[N_k(ST_{k-1}^j) \cup ST_{k-1}^j]$. By Lemma 21, $T_k^k[N_k(ST_{k-1}^j) \cup ST_{k-1}^j]$ is a $(k-j)$-dimensional partition tree $PT_{k-j}$. As a sequence, $T_k^k$ is isomorphic to the tree rooted at $u$ that has $v$ as a child of $u$ and $v$ has $k-1$ subtrees $PT_1, PT_2, \ldots,$ and $PT_{k-1}$ (see Fig. 10(b)).

So far, we have proven that the theorem also holds for $n = k$. Hence, the theorem holds. □

### 4.2. Height of independent spanning trees

In order to analyze the height of independent spanning trees, we first give the following lemma.

**Lemma 25.** *For any integer $n$ with $n \ge 1$, Height $(PT_n) = n$.*

**Proof.** We will prove the lemma by induction on $n$. The lemma clearly holds for $n = 1$. Supposing that the lemma holds for $n = k-1$ $(k \ge 2)$, we consider $n = k$. By Definition 19, Height $(PT_k) =$ Height$(PT_{k-1}) + 1$. By the induction hypothesis, Height$(PT_{k-1}) = k-1$. Therefore, Height $(PT_k) = k$. □

**Theorem 26.** *For any integer $n$ with $n \ge 2$, the height of each independent spanning tree on $TQ_n$ constructed by the proof of Theorem 17 is $n+1$.*

**Proof.** For $n = 2$, it is obvious that the theorem holds. For $n \ge 3$, by Theorem 24, all independent spanning trees are isomorphic to the graph for $n = k$ shown in Fig. 10(b). Furthermore, by Lemma 25, Height $(PT_{n-1}) = n-1$. Thus the height of each independent spanning tree on $TQ_n$ constructed by the proof of Theorem 17 is Height $(PT_{n-1}) + 2 = n+1$. □

## 5. Algorithm TQIST and its analysis

Based on the proof of Theorem 17, we provide an algorithm to generate $n$ independent spanning trees rooted at any vertex $v$ on $TQ_n$. The algorithm can be describes as follows.

### 5.1. Algorithm TQIST

**Algorithm 1** TQIST

**Input:** A positive integer $n$ and $v \in V(TQ_n)$
**Output:** A set of $n$ independent spanning trees rooted at vertex $v$.

```
BEGIN
  if n = 1 then
    T_1^1 = ({v, N_1(v)}, {(v, N_1(v))}); ST_1^1 = {N_1(v)};
  else
    call TQIST (n − 1, v);
  end if
  ST_n^n = N_n(v);
  parent(N_n(v), T_n^n) = v;
  for 1 ≤ i ≤ n − 1 do
    construct T_{n−1}^{1,i} according to Rule 9; ST_n^i = ST_{n−1}^i ∪ N_n(ST_{n−1}^i);
    V(T_n^i) = V(T_{n−1}^{0,i}) ∪ V(T_{n−1}^{1,i}); E′ = {(u, N_n(u))|u ∈ ST_{n−1}^i};
    if v ∈ V(TQ_{n−1}^0) then
      for each vertex u ∈ ST_{n−1}^i do
        parent(N_n(u), T_n^i) = u;
      end for
      E′′ = E(T_{n−1}^{1,i}[N_n(ST_{n−1}^i)]); E(T_n^i) = (E(T_{n−1}^{0,i}) ∪ E(T_{n−1}^{1,i}) ∪ E′) − E′′;
      for each vertex u ∈ N_n(ST_{n−1}^i) do
        parent(u, T_n^n) =parent (u, T_{n−1}^{1,i});
      end for
      for each vertex w ∈ ST_{n−1}^i do
        parent(w, T_n^n) = N_n(w);
      end for
    else
      for each vertex u ∈ ST_{n−1}^i do
        parent(u, T_n^i) = N_n(u)
      end for
      E′′ = E(T_{n−1}^{0,i}[ST_{n−1}^i]); E(T_n^i) = (E(T_{n−1}^{0,i}) ∪ E(T_{n−1}^{1,i}) ∪ E′) − E′′;
      for each vertex u ∈ ST_{n−1}^i do
        parent(u, T_n^n) =parent (u, T_{n−1}^{0,i});
      end for
      for each vertex w ∈ N_n(ST_{n−1}^i) do
        parent(w, T_n^n) = N_n(w);
      end for
    end if
  end for
END
```

### 5.2. Time complexity of Algorithm TQIST

Based on Algorithm TQIST, in the following, we will analyze the time complexity of Algorithm TQIST.

**Theorem 27.** *Algorithm TQIST constructs $n$ independent spanning trees on $TQ_n$ in $O(N \log N)$ time, where $N = 2^n$ is the number of vertices in $TQ_n$.*

**Proof.** Based on Algorithm TQIST, let $f(N)$ denote the running time of Algorithm TQIST. Then we have a recurrence equation that bounds $f(N)$ : $f(N) = 2f(N/2) + cN$, where $c$ is a constant. Solve this recurrence equation, then we can obtain $f(N) = O(N \log N)$. □

## 6. Conclusions

There are two versions for the $n$ independent spanning trees conjecture. The vertex (edge) conjecture is that any $n$-connected ($n$-edge-connected) graph has $n$ vertex-independent (edge-independent) spanning trees rooted at an arbitrary vertex. It has been proven that the vertex conjecture implies the edge conjecture. In this paper, we propose a construction method to construct $n$ isomorphic vertex-independent spanning trees rooted at any vertex on the $n$-dimensional twisted cube and the height of each independent spanning tree is $n+1$ for any integer $n$ with $n \ge 2$. In addition, we provide an $O(N \log N)$ algorithm to construct $n$ independent spanning trees rooted at any vertex on $TQ_n$, where $N$ denotes the number of vertices in $TQ_n$.

## Acknowledgments

We would like to express our warmest gratitude to Professor Guoliang Chen for his constructive suggestions and the anonymous referees for their review comments that helped improve the

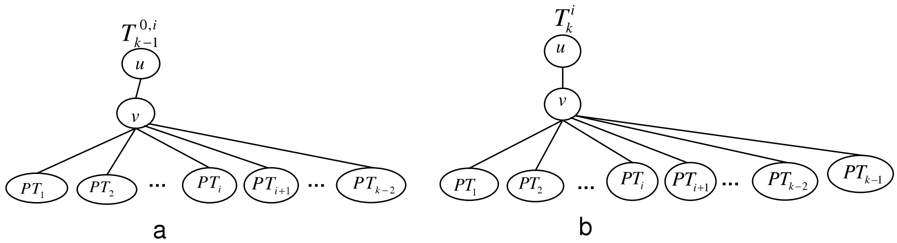



**Fig. 10.** (a) An illustration of $T_{k-1}^{0,i}$. (b) An illustration of $T_k^i$.

original manuscript. This work is supported by the National Natural Science Foundation of China (60873047, 61170021 and 60970117), the Natural Science Foundation of Jiangsu Province (No. BK2008154), the Specialized Research Fund for the Doctoral Program of Higher Education (No. 20103201110018), the Hong Kong RGC grant (CityU 7008041) and sponsored by the Qing Lan Project.

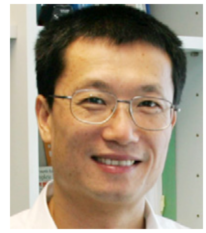
## References

[1] S. Abraham, K. Padmanabhan, The twisted cube topology for multiprocessors: a study in network asymmetry, Journal of Parallel and Distributed Computing 13 (1) (1991) 104–110.
[2] F. Bao, Y. Igarashi, K. Katano, Broadcasting in hypercubes with randomly distributed Byzantine faults, in: Distributed Algorithms, 1995, pp. 215–229.
[3] F. Bao, Y. Igarashi, S.R. öhring, Reliable broadcasting in product networks, Discrete Applied Mathematics 83 (1–3) (1998) 3–20.
[4] C.P. Chang, J.N. Wang, L.H. Hsu, Topological properties of twisted cube, Information Sciences 113 (1–2) (1999) 147–167.
[5] J. Cheriyan, S.N. Maheshwari, Finding nonseparating induced cycles and independent spanning trees in 3-connected graphs, Journal of Algorithms 9 (4) (1988) 507–537.
[6] S. Curran, O. Lee, X. Yu, Finding four independent trees, SIAM Journal on Computing 35 (5) (2006) 1023–1058.
[7] K. Efe, A variation on the hypercube with lower diameter, IEEE Transactions on Computers 40 (11) (1991) 1312–1316.
[8] J. Fan, X. Jia, B. Cheng, J. Yu, An efficient fault-tolerant routing algorithm in bijective connection networks with restricted faulty edges, Theoretical Computer Science 412 (29) (2011) 3440–3450.
[9] J. Fan, X. Jia, X. Lin, Optimal embeddings of paths with various lengths in twisted cubes, IEEE Transactions on Parallel and Distributed Systems 18 (4) (2007) 511–521.
[10] J. Fan, X. Jia, X. Lin, Embedding of cycles in twisted cubes with edge-pancyclic, Algorithmica 51 (3) (2008) 264–282.
[11] J. Fan, X. Jia, X. Liu, S. Zhang, J. Yu, Efficient unicast in bijective connection networks with the restricted faulty node set, Information Sciences 181 (11) (2011) 2303–2315.
[12] J. Fan, X. Lin, The t/k-diagnosability of the BC graphs, IEEE Transactions on Computers 54 (2) (2005) 176–184.
[13] J. Fan, X. Lin, Y. Pan, X. Jia, Optimal fault-tolerant embedding of paths in twisted cubes, Journal of Parallel and Distributed Computing 67 (2) (2007) 205–214.
[14] P. Fragopoulou, S.G. Akl, Edge-disjoint spanning trees on the star network with applications to fault tolerance, IEEE Transactions on Computers 45 (2) (1996) 174–185.
[15] T. Hasunuma, H. Nagamochi, Independent spanning trees with small depths in iterated line digraphs, Discrete Applied Mathematics 110 (2–3) (2001) 189–211.
[16] S.Y. Hsieh, C.J. Tu, Constructing edge-disjoint spanning trees in locally twisted cubes, Theoretical Computer Science 410 (8–10) (2009) 926–932.
[17] S.Y. Hsieh, Y.F. Weng, Fault-tolerant embedding of pairwise independent Hamiltonian paths on a faulty hypercube with edge faults, Theory of Computing Systems 45 (2) (2009) 407–425.
[18] S.Y. Hsieh, C.Y. Wu, Edge-fault-tolerant Hamiltonicity of locally twisted cubes under conditional edge faults, Journal of Combinatorial Optimization 19 (1) (2010) 16–30.
[19] S.Y. Hsieh, P.Y. Yu, Fault-free mutually independent Hamiltonian cycles in hypercubes with faulty edges, Journal of Combinatorial Optimization 13 (2) (2007) 153–162.
[20] W.T. Huang, J.J.M. Tan, C.N. Hung, L.H. Hsu, Fault-tolerant Hamiltonicity of twisted cubes, Journal of Parallel and Distributed Computing 62 (4) (2002) 591–604.
[21] A. Huck, Independent trees in graphs, Graphs and Combinatorics 10 (1) (1994) 29–45.
[22] A. Huck, Independent trees in planar graphs, Graphs and Combinatorics 15 (1) (1999) 29–77.
[23] A. Itai, M. Rodeh, The multi-tree approach to reliability in distributed networks, in: Proceedings of the 25th Annual Symposium on Foundations of Computer Science, 1984, pp. 137–147.
[24] S.L. Johnsson, C.T. Ho, Optimum broadcasting and personalized communication in hypercubes, IEEE Transactions on computers 38 (9) (1989) 1249–1268.
[25] S. Khuller, B. Schieber, On independent spanning trees, Information Processing Letters 42 (6) (1992) 321–323.
[26] C.J. Lai, C.H. Tsai, Embedding a family of meshes into twisted cubes, Information Processing Letters 108 (5) (2008) 326–330.
[27] P.L. Lai, C.H. Tsai, Embedding of tori and grids into twisted cubes, Theoretical Computer Science 411 (40–42) (2010) 3763–3773.
[28] J.-C. Lin, J.-S. Yang, C.-C. Hsu, J.-M. Chang, Independent spanning trees vs. edge-disjoint spanning trees in locally twisted cubes, Information Processing Letters 110 (10) (2010) 414–419.
[29] Y.J. Liu, J.K. Lan, W.Y. Chou, C. Chen, Constructing independent spanning trees for locally twisted cubes, Theoretical Computer Science 412 (22) (2011) 2237–2252.
[30] K. Miura, D. Takahashi, S. Nakano, T. Nishizeki, A linear-time algorithm to find four independent spanning trees in four-connected planar graphs, in: Graph-Theoretic Concepts in Computer Science, 1998, pp. 115–132.
[31] S. Nagai, S.I. Nakano, A linear-time algorithm to find independent spanning trees in maximal planar graphs, IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences E84A (5) (2001) 1102–1109.
[32] K. Obokata, Y. Iwasaki, F. Bao, Y. Igarashi, Independent spanning trees of product graphs and their construction, IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences E79-A (11) (1996) 1894–1903.
[33] P. Ramanathan, K.G. Shin, Reliable broadcast in hypercube multicomputers, IEEE Transactions on Computers 37 (12) (1988) 1654–1657.
[34] S. Tang, Y.L. Wang, Y.H. Leu, Optimal independent spanning trees on hypercubes, Journal of Information Science and Engineering 20 (1) (2004) 143–155.
[35] M.C. Yang, Constructing edge-disjoint spanning trees in twisted cubes, Information Sciences 180 (20) (2010) 4075–4083.
[36] X. Yang, Q. Dong, E. Yang, J. Cao, Hamiltonian properties of twisted hypercube-like networks with more faulty elements, Theoretical Computer Science 412 (22) (2011) 2409–2417.
[37] X. Yang, D.J. Evans, G.M. Megson, The locally twisted cubes, International Journal of Computer Mathematics 82 (4) (2005) 401–413.
[38] J.S. Yang, S.M. Tang, J.M. Chang, Y.L. Wang, Parallel construction of optimal independent spanning trees on hypercubes, Parallel Computing 33 (1) (2007) 73–79.
[39] A. Zehavi, A. Itai, Three tree-paths, Journal of Graph Theory 13 (2) (1989) 175–188.

**Yan Wang** received the B.S. and M.S. degrees from the Nanjing University of Aeronautics and Astronautics in 1999 and 2002, respectively. She is currently a Ph.D. candidate in Computer Science at the Soochow University. Her research interests include parallel and distributed systems, algorithms.

**Jianxi Fan** received the B.S., M.S., and Ph.D. degrees in Computer Science from the Shandong Normal University, Shandong University, and the City University of Hong Kong, China, in 1988, 1991, and 2006, respectively. He is currently a Professor of Computer Science with the School of Computer Science and Technology at the Soochow University, China. He visited the Department of Computer Science at the City University of Hong Kong, Hong Kong (October 2006–March 2007, June 2009–August 2009) as a research fellow. His research interests include parallel and distributed systems, interconnection architectures, design and analysis of algorithms, and graph theory.

**Guodong Zhou** received the Ph.D. degree from the National University of Singapore in 1999. He joined the Institute for Infocomm Research, Singapore, in 1999, and had been the associate scientist, scientist and associate lead scientist at the institute until August 2006. Currently, he is a distinguished Professor at the School of Computer Science and Technology, Soochow University, Suzhou, China. His research interests include natural language processing, information extraction and machine learning, and algorithms. He is a senior member of the CCF and has been the member of ACM and IEEE since 1999. Currently, he serves as an editorial board member of Computational Linguistics and an associate editor of ACM Transaction on Asian Language Information Processing.



**Xiaohua Jia** received his B.Sc. (1984) and MEng (1987) from the University of Science and Technology of China, and DSc (1991) in Information Science from the University of Tokyo. He is currently the Chair Professor with the Dept of Computer Science at the City University of Hong Kong. His research interests include distributed systems, computer networks, wireless sensor networks and mobile wireless networks. Prof. Jia is an editor of IEEE Trans. on Parallel and Distributed Systems (2006–2009), Wireless Networks, Journal of World Wide Web, Journal of Combinatorial Optimization, etc. He is the General Chair of ACM MobiHoc 2008, the TPC Co-Chair of IEEE MASS 2009, the Area-Chair of IEEE INFOCOM 2010, the TPC Co-Chair of IEEE GlobeCom 2010—Ad Hoc and Sensor Networking Symp, and the Panel Co-Chair of IEEE INFOCOM 2011.